Koszul duality in the hypertoric case

OK, so, we've chosen a torus K; if we consider reps of $U_m = D^K / \mathrm{Ker}(m : U(\mathfrak{k}) \to \mathbb{C})$ with integral weights, then generically we'll get a category equivalent to the A-algebra of the induced hyperplane arrangement in $\eta + V^* \subset \mathbb{C}^n$ by intersection w/ the coordinate hyperplanes.

$$\text{chambers} \longleftrightarrow \begin{array}{l}\text{simples / projectives}\\ \text{corresponding to integral}\\ \text{weights in the chamber}\end{array}$$

On the other hand, if we compute the Coulomb branch of K w/ $\mathbb{C}^n$ as matter, then we get the A-algebra of the induced hyperplane arrangement on $\mathfrak{k} + \xi$; call this B to avoid confusion.

What's the relationship between these? We know that both are Koszul.

Every chamber in $\eta + V^*$ is a linear program, and its dual is a chamber in $\mathfrak{k} + \xi$, with objective $-\eta$.

This program is feasible and bounded if and only if its ~~d~~ dual is. Thus the simples / projectives in these two category $\mathcal{O}$'s are in canonical bijection.

Thus $A_0 \cong B_0$.

A_1 corresponds to pairs of chambers that differ by a single sign. Thus, the bijection preserves adjacencies.

We can take $A_1 \cong B_1$, but actually I want a duality between them.

We have to be a little careful about signs here: we call a pair of chambers odd if there are an odd number of ~~signs~~ + -signs in $\varepsilon_1, \dots, \varepsilon_{i-1}$ if the differ in ε_i. This has the property that any square has an odd number of odd signs.

We pair A_1 and B_1 by

$$\langle(\alpha,\beta),(\gamma,\delta)\rangle = \begin{cases} 0 & \text{unless } \alpha \neq \gamma \text{ and } \beta \neq \delta \\ 1 & \text{if } \alpha = \gamma \text{ and } \beta = \delta \text{ and } (\alpha,\beta) \text{ is even} \\ -1 & \text{if } \alpha = \gamma \text{ and } \beta = \delta \text{ and } (\alpha,\beta) \text{ is odd} \end{cases}$$

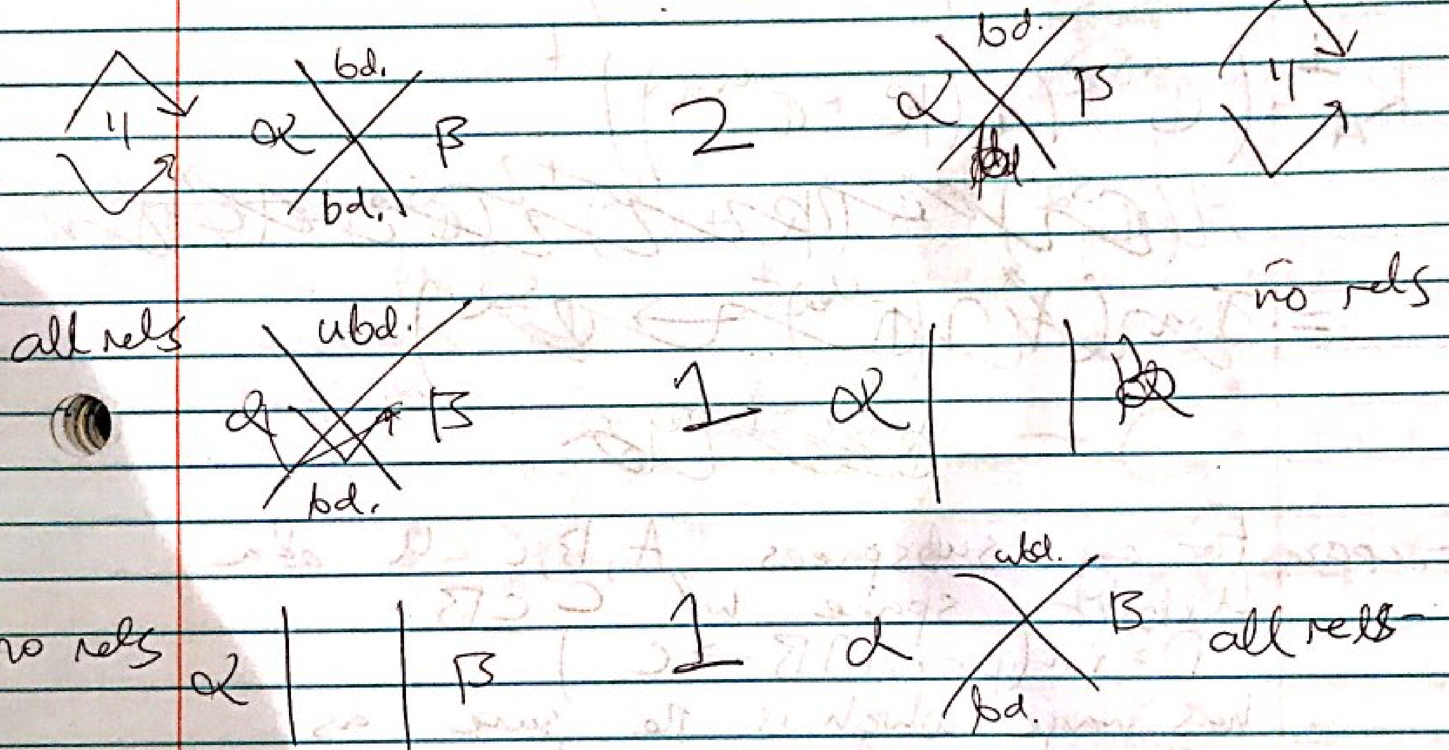
So, we need to check that this induces a quadratic duality: if $R_A \subseteq A_1 \otimes_{A_0} A_1$, $R_B \subseteq B_1 \otimes_{B_0} B_1$

then $R_A^\perp = R_B$ and vice versa

We can check this for each pair α, β of chambers that differ in 2 or 0 places for other chambers. There are 4 possibilities to consider:

primal | dual

$\alpha = \beta \quad \dim_{e_\beta} A, \varphi_{A_0}, A, e_\alpha \quad \alpha^* = \beta^*$



Finally need to consider $\alpha = \beta$

$\dim_{e_\alpha} A, \Theta_{A_0}, A_{e_\alpha}$ = # single sign changes from α to a b.f. ~~class~~ sign vector. Let S_α be the set of positions where these sign changes happen and U_α the set of changes ~~that~~ which send α to an unbounded but feasible sign vector, and I_α the ~~infeasible~~ changes that flip to an infeasible but bounded chamber

Lemma $[1,n] = S_\alpha \cup U_\alpha \cup I_\alpha$

The relations are given by

$$K_A = \mathbb{C}^{S_\alpha} \cap (R + \mathbb{C}^{U_\alpha})$$

And in the algebra B by

$$K_B = \mathbb{C}^{S_\alpha} \cap (t^* + \mathbb{C}^{I_\alpha}).$$

$$K_A^\perp = \left(\mathbb{C}^{S_\alpha} \cap (R + \mathbb{C}^{U_\alpha})\right)^\perp$$

$$\cancel{A((\mathbb{C}^{S_\alpha})^\perp + (R + \mathbb{C}^{U_\alpha})^\perp = (\mathbb{C}^{S_\alpha})^\perp + (t^* \cap (\mathbb{C}^{U_\alpha})^\perp))}$$

$$= \cancel{\mathrm{Im}(t^* \cap (\mathbb{C}^{U_\alpha})^\perp \to \mathbb{C}^{S_\alpha})}$$

$$\cancel{\mathbb{C}^{S_\alpha} + \mathbb{C}^{I_\alpha}}$$

Exercise For any subspaces $A, B, C \subset \mathcal{H}$ of a Hilbert space w/ $C \subseteq B$

$D = \mathrm{im}(\mathrm{pr}: A \cap B \to C)$

has image which is the same as projecting A to $B^\perp + C$ and intersecting w/ C. $\Leftrightarrow D = (A + (B \cap C^\perp)) \cap C$

Apply this with $C = \mathbb{C}^{S_\alpha}$, $B = \mathbb{C}^{S_\alpha} + \mathbb{C}^{I_\alpha}$, $A = t^*$

$\cancel{K_A^\perp = t^{*\perp}}$

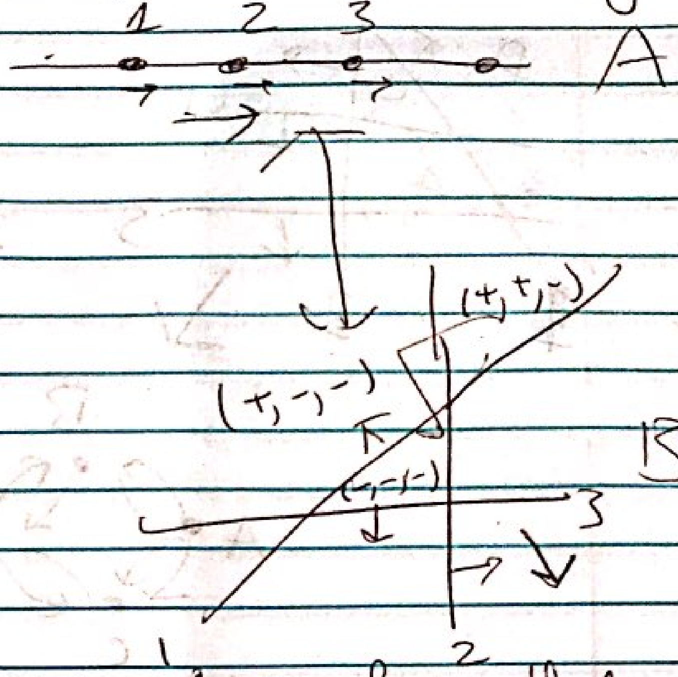
$$C^\perp \cap B = \mathbb{C}^{I_\alpha}$$

$$K_B = (t^* + \mathbb{C}^{I_\alpha}) \cap \mathbb{C}^{S_\alpha} = \mathrm{im}(\mathrm{pr}\ t^* \cap (\mathbb{C}^{S_\alpha + I_\alpha}) \to \mathbb{C}^{S_\alpha})$$

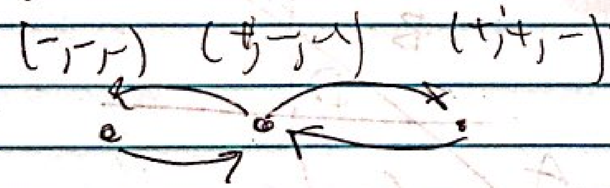
$$= (\mathbb{C}^{S_\alpha} \cap (R + \mathbb{C}^{U_\alpha}))^\perp = K_A^\perp$$

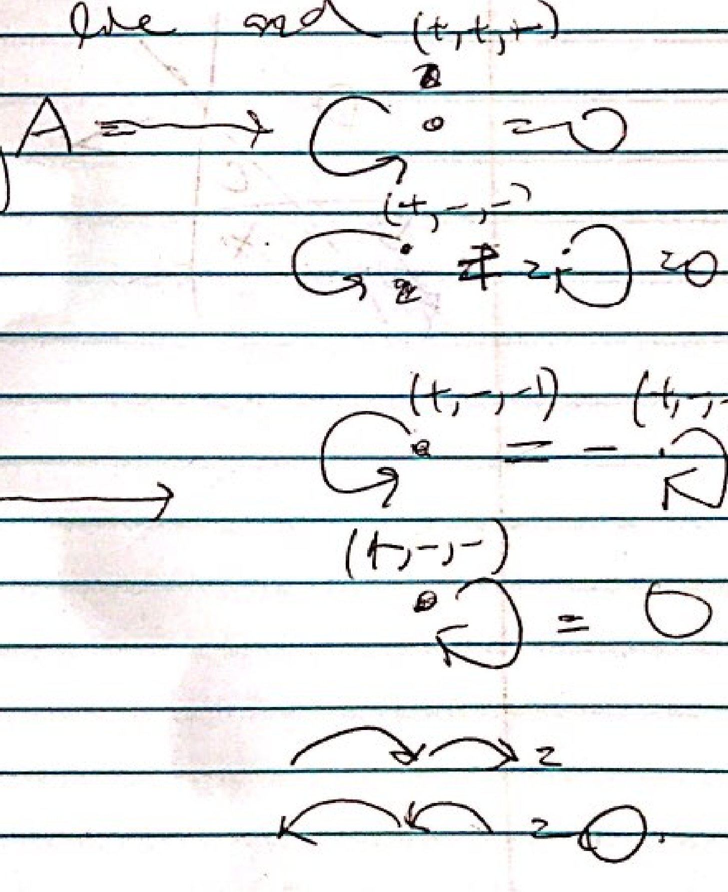
This proves the Koszul duality.

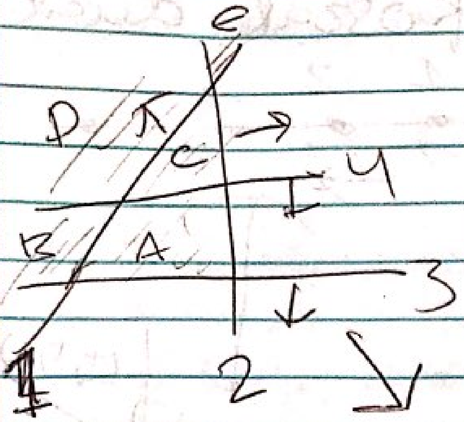
For example:



In both cases, the feasible chambers are in a line and the quiver is



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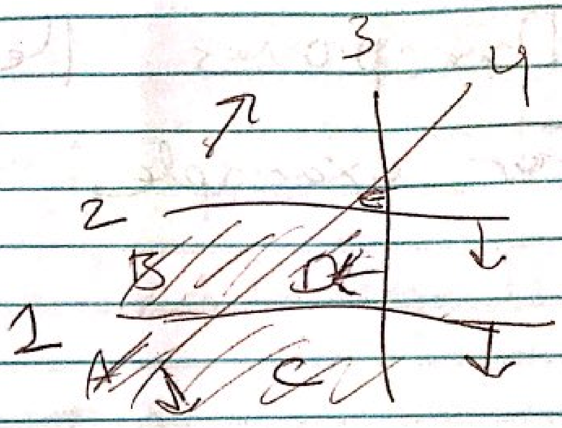
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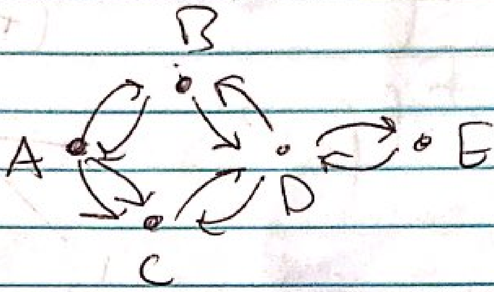
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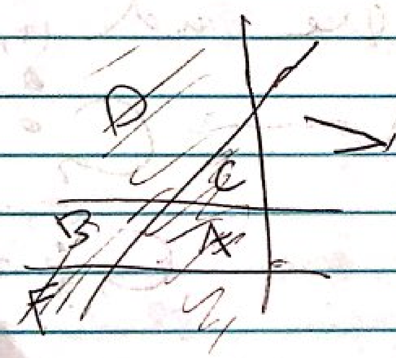
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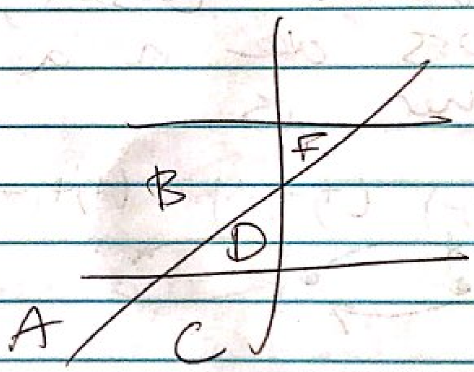
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