Koszul Duality I | **Def:** A *positively graded alg.* $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is a grdd algebra (over a base field $\mathbb{F}$) s.t. • $A_i = 0 \ \forall i < 0$ • $\dim_{\mathbb{F}} A_i < \infty \ \forall i$

• $A_0 \equiv \mathbb{k}$ is semisimple.

Ex 1: Let V be a fid v.s. $/\mathbb{F}$. The follwing algs: $(\mathbb{k} = \mathbb{F})$

• $T = T^* V = \mathbb{k}\langle x_1, \dots, x_n \rangle$ where $\{x_i\}$ basis for V. $\deg x_i = 1$.

• $S = S^* V = \mathbb{k}[x_1, \dots, x_n] = T / _{x_i x_j = x_j x_i}$

• $\Lambda = \Lambda^* V = T / _{x_i x_j = -x_j x_i,\ x_i^2 = 0}$

Ex 2: Γ a quiver.
$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, "a"] & \overset{2}{\bullet} \arrow[r, bend left, "b"] & \overset{3}{\bullet} \arrow[l, bend left, "c"] \end{tikzcd}$$
$A = \mathbb{F}[\Gamma]$ where A_i has basis given by paths of length i. $A_0 = \mathbb{k} = \mathrm{Span}\{e_1, e_2, e_3\} \cong \mathbb{F} \times \mathbb{F} \times \mathbb{F}$

$A_1 = \mathrm{Span}\{a, b, c\}$ $\quad A_2 = \mathrm{Span}\{ba, bc, cb\}$ etc.

Ex 3: Quiver alg w/ homog relns, like $cb = 0$.

All the above examples have something in common: they are quadratic.

Def: A is *quadratic* if generated over $\mathbb{k}$ by $A_1 = V$, w/ relns in degree 2.

i.e. $A \cong T^*_{\mathbb{k}} V / (R)$ $\quad R \subset V \otimes_{\mathbb{k}} V$ $\quad$ (Rmk: V is a $\mathbb{k}$-bimodule.)

Def: Given any pga A, get a quadratic alg $A^!$ as follows:

let $m : A_1 \otimes A_1 \to A_2$, take adjoint $m^* : A_2^* \to (V \otimes V)^* = V^* \otimes V^*$
$\quad\ \ \overset{\shortparallel}{V \otimes V}$

$\left(\begin{array}{l} \text{Rmk: } (\ )^* \text{ means } \mathrm{Hom}_{\mathbb{k}}(-, \mathbb{k}) \\ (V \otimes W)^* = W^* \otimes V^* \text{ canonically.} \end{array} \right)$

Let $R^! = \mathrm{Im}(m^*)$. Then set $A^! = T^* V^* / (R^!)$

Ex: S for $\dim V = 2$. $\quad m : V \otimes V \to S_2$

$V = \mathrm{Span}\{x_1, x_2\}$

$S_2 = \mathrm{Span}\{x_1^2, x_1 x_2, x_2^2\}$

$$m = \begin{array}{c} \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 \otimes x_1 & x_1 \otimes x_2 & & \end{array} \\ \begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & & & 1 \end{pmatrix} \end{array} \qquad m^* = \begin{pmatrix} 1 & & \\ & 1 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

image is $R^! = \mathrm{Span}\begin{pmatrix} x_1 \otimes x_1 \\ x_1 \otimes x_2 + x_2 \otimes x_1 \\ x_2 \otimes x_2 \end{pmatrix}$ $\Rightarrow S^! = \Lambda.$

<u>Exercise</u>: $A_Q = \Bbbk\left[\underset{1}{\circ} \overset{f}{\underset{g}{\rightleftarrows}} \underset{2}{\circ}\right] \Big/ fg=0$. Then $A_Q^! = \Bbbk\left[\circ \overset{f^\vee}{\underset{g^\vee}{\rightleftarrows}} \circ\right] \Big/ g^\vee f^\vee = 0$

<u>Rmk</u>: Spose A is quadratic. Then $A_2 = V\otimes V/R$ $m: V\otimes V \longrightarrow V\otimes V/R$

then m^* has image $R^\perp \subset (V\otimes V)^*$. $R^! = R^\perp$.

Since $(R^\perp)^\perp = R$ we have $(A^!)^! = A$. So $A^!$ is called <u>quadratic dual</u>.

Any pga has left module $\Bbbk = A/A_+$. What does a ~~free~~ minimal proj resolution of $\Bbbk$ look like?

$$\text{etc} \longrightarrow \bigoplus A\langle -r_j\rangle \xrightarrow{\Delta} \bigoplus A\langle -d_i\rangle \xrightarrow{\varphi} A \twoheadrightarrow \Bbbk \to 0$$

φ surjects onto generators for A_+. Choose generators $\{x_i\}$ of A w/ degrees d_i.

Δ surjects onto relations, in degrees r_j

etc are the relns b/w relns, a "higher presentation"

<u>Ex</u>: S, $\dim V = 2$

$$\left(S\langle -2\rangle \xrightarrow{\begin{bmatrix} y \\ -x\end{bmatrix}} \begin{matrix} S\langle -1\rangle \\ \oplus \\ S\langle -1\rangle \end{matrix} \xrightarrow{[x\ y]} S \right) \to \Bbbk \to 0$$

$dx\,dy$ $\quad dx, dy \quad 1$

<u>Ex</u>: $S = T$, $\dim V = 1$

$$0 \to S\langle -1\rangle \xrightarrow{x} S \to \Bbbk \to 0$$

$dx \qquad 1$

<u>Ex</u>: Λ, $\dim V = 1$
$= \Bbbk[x]/_{x^2}$

$$\cdots \xrightarrow{x} \Lambda\langle -3\rangle \xrightarrow{x} \Lambda\langle -2\rangle \xrightarrow{x} \Lambda\langle -1\rangle \xrightarrow{x} \Lambda \to \Bbbk \to 0$$

$x^3 \quad x^2 \quad x \quad 1$

<u>Ex</u> Λ, $\dim V = 2$

$$\begin{matrix} \Lambda_{x^2} \\ \oplus \\ \Lambda_{xy} \\ \oplus \\ \Lambda_{y^2} \end{matrix} \to \begin{matrix} \Lambda_x \\ \oplus \\ \Lambda_y \end{matrix} \to \Lambda \to \Bbbk$$

a) Quadratic $\iff$ hom. degree $0, -1, -2$ parts have shifts $0, -1, -2$ respectively

<u>Observations</u>: 1) In each case the entire complex is <u>linear</u>: in hom deg i, have grading shift i

(Call differentials are "degree 1")

this will be <u>Koszul</u>.

2) The "size" of the resolution seems to agree w/ "size" of the dual $A^!$. Called above

<u>Koszul complex</u>

Def: A a pga is Koszul if $\Bbbk$ has a linear projective resolution.

Rmk/Ex: Why proj and not free? When $\Bbbk = \mathbb{F}$ no difference. Any projective up to shift is a summand of A, has form Ae for $e \in \Bbbk$ idempotent. Lin: $Ae\langle i\rangle$ only appears in hom deg i.

Ex: A_Q. Naive:

$$A \xrightarrow{\begin{bmatrix} 0 \\ f \end{bmatrix}} \begin{matrix} A\langle -1\rangle \\ \oplus \\ A\langle 1\rangle \end{matrix} \xrightarrow{[f, g]} A \to \Bbbk$$

b/c only thing that kills f is $\cdot f$, nothing kills $\cdot f$ (fg does, but already zero)

but this isn't quite true — $e_1 f = 0$ as well... This would give deg 0 differential...

Correct:

$$0 \to Ae_1 \xrightarrow{\begin{bmatrix} 0 \\ f \end{bmatrix}} \begin{matrix} Ae_1 \\ \oplus \\ Ae_2 \end{matrix} \xrightarrow{[f, g]} A \to \Bbbk$$

This is exact.

Rmk: Koszul $\Rightarrow$ Quadratic $\not\Leftarrow$ Example.

Rmk: General nonsense: if $P^\bullet \to M$, $Q^\bullet \to M$ are two proj resns then

$$\begin{matrix} P^\bullet & & \\ f \downarrow & \searrow & M \\ Q^\bullet & \nearrow & \end{matrix}$$

f a hom eq. unique up to homotopy

if P, Q are linear, then there are no homotopies!! $A\langle -1\rangle$, $A\langle -2\rangle$ no maps of neg degree!

$\Rightarrow f$ an isom of chain complexes, unique!

linear proj resns are canonical (when they exist)

Can use this to show differentials are $\Bbbk$-bimodule maps (not just left module)

Def: A a pga. Then the "Koszul dual category" (to A-gmod) $\mathcal{LP}(A)$ has

Ob: linear complexes of projectives. Mor: Chain maps (no homotopies anyway)

Prop: $\mathcal{LP}(A)$ is an abelian category!

Sketch: Any degree 0 map b/w projectives is split. $Ae\langle i\rangle \to Af\langle i\rangle$ has ker, im, coker all just another projective w/ shift $\langle i\rangle$ b/c $\Bbbk$ is semisimple. So $\mathrm{Ker}(P^\bullet \to Q^\bullet)$ is an object of $\mathcal{LP}(A)$, etc.

Also graded: $P^\bullet\{1\} = P^\bullet\langle -1\rangle[1]$ still in $\mathcal{LP}(A)$.

Thm: A generated in deg 1 then $\mathcal{LP}(A) \cong A^!$-gmod
$\{1\} \leftrightarrow \langle 1\rangle$.

Idea: Let $X=\bigoplus X_i$ a graded $\mathbb{k}$-bimod. Then
$\{$ways to equip X w/ $A^!$-gmod structure$\}$ $\Longleftrightarrow$ $\{$ways to put an A-linear differential on $\cdots \to A\overset{\langle i\rangle}{\otimes_{\mathbb{k}}} X_i \xrightarrow{d} A\overset{\langle i+1\rangle}{\otimes_{\mathbb{k}}} X_{i+1} \to \cdots$ w/ $d^2=0$

i.e. size of complex (eg. mult spaces) is an $A^!$-gmod

Explicitly, given $X\in A^!$-gmod, let $d(a\otimes x) = \sum a\alpha_i\otimes\beta_i x$ where $\{\alpha_i\}\{\beta_i\}$ dual bases for V, V^*
$\beta_i \in A^!_1$, $x \in X_i$, $\alpha_i \in A_1$, $\beta_i x \in X_{i+1}$

Exercise: Indep of dual bases, $d^2=0$.

Conversely, $P^\bullet = A\otimes X$. How to act by $\beta\in V^*$ on $x\in X_i$?

$1\otimes x \xrightarrow{d} \sum a_j\otimes y_j$ so $\beta(x) = \sum \beta(a_j)\cdot y_j \in X_{i+1}$
$a_j \in V=A_1$, $y_j \in X_{i+1}$ for degree reasons

Exercise: relations $R^!$ are satisfied. Exercise: Inverse functors. Exer: Grading shift

Example: $\Lambda \circlearrowright \Lambda$, $S^! = \Lambda$, $X = \Lambda$
$S\otimes X_0 \longrightarrow S\overset{\langle 1\rangle}{\otimes} X_1$, $X_0 = \mathbb{k}\cdot 1$, $X_1 = \mathbb{k}\cdot dx$, $\{x\}\{dx\}$ dual bases
$1\otimes 1 \longmapsto x\otimes dx$ is just $S\xrightarrow{x} S\langle 1\rangle$ resolving $\mathbb{k}\langle 1\rangle[-1]$.

Example: $0\to A\to 0 \in \mathcal{LP}(A) \longleftrightarrow \mathbb{k}\circlearrowleft A^!$

When A is Koszul, it has $P^\bullet\in\mathcal{LP}(A)$, resolution of $\mathbb{k}$. What $A^!$-gmod is it?

$A^!$ is an $A^!$-bimod, $\mathbb{k}$-bimod. Then ${}^*A^! = \mathrm{Hom}_{(-,\mathbb{k})}(A^!,\mathbb{k})$ still has left $A^!$-action.

This reverses the grading: $({}^*A^!)_{-1} = (A^!_1)^*$

This leads to the Koszul complex, an explicit version of a complex

$$\cdots \to A\otimes {}^*A^!_2 \to A\otimes {}^*A^!_1 \to A\otimes {}^*A^!_0$$

What is ${}^*A^!_{\bullet}$?

$$A^!_0 = k \qquad {}^*A^!_0 = k$$

$$A^!_1 = V^* \qquad {}^*A^!_1 = V \qquad \text{so complex ends}$$

$$A^!_2 = V^*\otimes V^*/R^\perp \qquad {}^*A^!_2 = R \qquad \to A\otimes_k R^{\langle -2\rangle} \to A\otimes_k V^{\langle -1\rangle} \to A$$

so far so good

$$A^!_3 = V^*\otimes V^*\otimes V^* \big/ \left(R^\perp\otimes V^* + V^*\otimes R^\perp\right) \qquad {}^*A^!_3 = R\otimes V \cap V\otimes R \subset V\otimes V\otimes V$$

in general, $${}^*A^!_n = \bigcap \underline{V\otimes V\otimes \ldots \otimes R\otimes \ldots \otimes V} \subset V^{\otimes n}$$

Need a differential

$$A\otimes {}^*A^!_n\langle -n\rangle \xrightarrow{d} A\otimes {}^*A^!_{n-1}\langle -n+1\rangle$$

d is induced by $$a\otimes(v_1\otimes\ldots\otimes v_n) \longmapsto av_1\otimes(v_2\otimes\ldots\otimes v_n)$$

Why $d^2=0$? $d^2(\;''\;) = av_1v_2\otimes(\quad)$ and $v_1v_2\in R$ so $v_1v_2=0$ in A.

(pure tensors rarely live in ${}^*A^!_n$ but that's the idea)

This defines a complex $K^\cdot(A)$ for any pga A.

<u>Thm</u>: A is Koszul $\iff$ $(K^\cdot \to k)$ is exact. (and so $A^!$ is also Koszul, since $K(A^!) = K(A)^*$ is exact too)

$\Leftarrow$ obvious since given linear res. of k.

$\Rightarrow$: see BGS.

<u>Exercise</u>: Deduce that $A^! \cong \mathrm{Ext}^*_A(k,k)^{op}$