Quiver presentation

Recall that we defined $P_\chi \cong \varprojlim U/U\mathfrak{m}_\chi^N$ w/ $\mathrm{Hom}(P_\chi, M) \cong M_\chi$.

More generally continuous homomorphisms pick out topological weight vectors. ~~(which are [illegible])~~

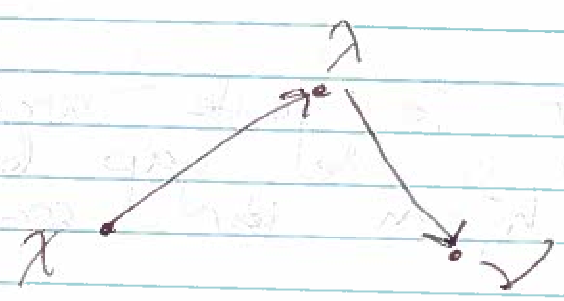
Thus $\mathrm{Hom}(P_\chi, P_\lambda) \cong (P_\lambda)_\chi$
$= \hat{S}\, r_{\lambda-\chi} v.$

Composition of these is fairly easy to calculate.

$$P_\chi \xrightarrow{v_\chi \mapsto f \cdot r_{\lambda-\chi} \cdot v_\lambda} P_\lambda \xrightarrow{v_\lambda \mapsto g\, r_{\nu-\lambda} \cdot v_\nu} P_\nu$$

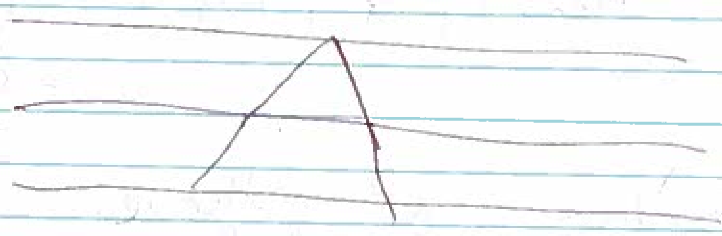
$$v_\chi \longmapsto fg\, r_{\lambda-\chi} r_{\nu-\lambda} \cdot v_\nu$$

Thus, when we calculate we can use the graphical multiplication.



For purposes of calculation, we draw in the hyperplanes where $h_i^{\pm}(X) \equiv h_i^{\pm}(\lambda) \bmod \mathbb{Z}$.

The terms in $r_{\lambda-\chi}\, r_{\nu-\lambda} / r_{\nu-\chi}$ are in bijection w/ hyperplanes we cross twice, or I imagine better to say gaps between hyperplanes.



this crossing two gaps.

It's easy to see the factors you get are $h_i^{\pm}$ corresponding to hyperplanes crossed with some shifts, but where are the shifts?

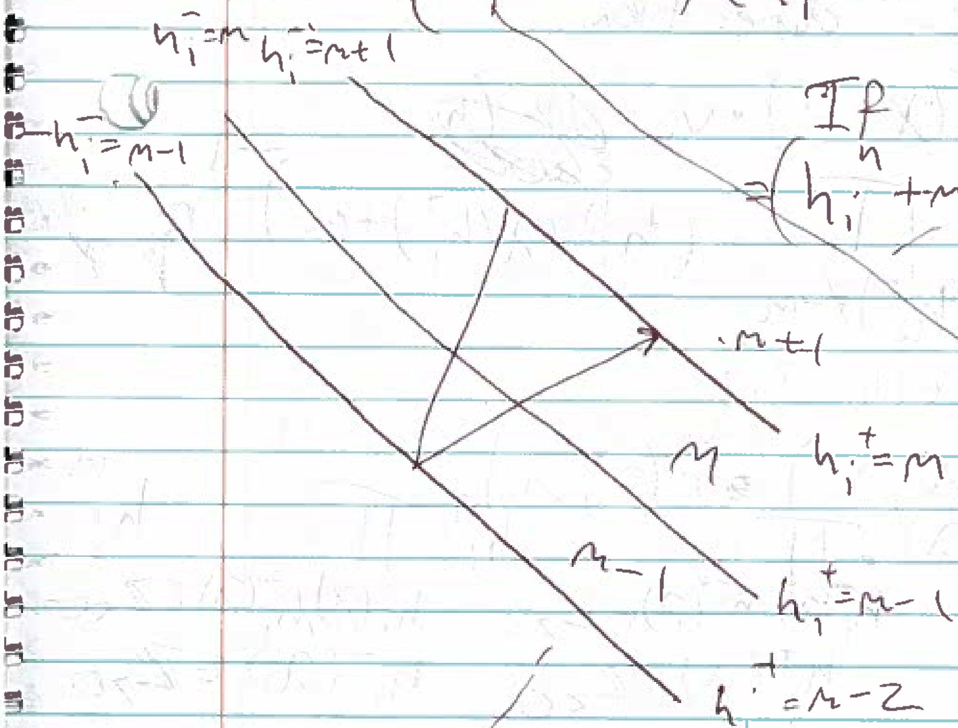
Easier to write in terms of S since then we don't need to worry about whether we multiply on the left or right. Since we've fixed the weight space h differs from $h_i^{\pm}$ by just a scalar.

The answer is quite nice! every gap is the gap between $h_i^+ = m$ and $h_i^- = m$ for some $m \in \mathbb{Z}$.

Undoing crossing this gap twice multiplies by $h_i^n + m$.

This is a pain to actually prove but it's worth thinking about how to check special cases. For simplicity, assume $m = \chi(h_i^+) = \nu(h_i^+)$. In this case, z_i it contributes a factor $[h_i]$

If $z_i > 0$, then we get $(h_i^- - 1)(\cdots(h_i^- - z_i)\cdots$
$= (h_i^n + m)(h_i^n + m - 1) \cdots (h_i^n + m - z_i + 1)$



If $z_i < 0$, $(h_i^+ + 1)\cdots$
$= (h_i^n + m + 1)(h_i^n + m + 2)\cdots$

So, for these terms are invertible, unless $m = 0$!

Idea: write morphisms

$\tilde{r}(\chi,\nu) : P_\nu \to P_\chi$

"Corrected" so that these [illegible] terms go away.

Since these terms ~~must appear~~ come from crossing a hyperplane twice, we need to [illegible] these terms for "half" the hyperplanes we cross. Luckily, hyperplanes are cooriented, so we can ask whether we cross in positive or negative direction.

Let $\tilde{r}(\chi,\nu)\cdot v_\nu =$

$$\prod_{\substack{i \\ z_i>0}} \ \prod_{\substack{\chi(h_i^+)\neq -m \\ m\in\{1,\dots,z_i\}}} \left(h_i^n + \chi(h_i^+) + m\right)^{-1} r_{\nu-\chi}$$

Cor $\tilde{r}(\chi,\nu)\,\tilde{r}(\nu,\lambda)$

$$= \tilde{r}(\chi,\lambda)\cdot \prod_{\substack{h_i^-(\chi),\, h_i^-(\lambda)\in\mathbb{Z}_{>0} \\ h_i^+(\lambda)\in\mathbb{Z}_{<0}}} h_i^n \ \prod_{\substack{h_i^+(\lambda),\, h_i^-(\lambda)\in\mathbb{Z}_{<0} \\ h_i^-(\lambda)\in\mathbb{Z}_{>0}}} h_i^n \qquad (*)$$

~~Cor~~

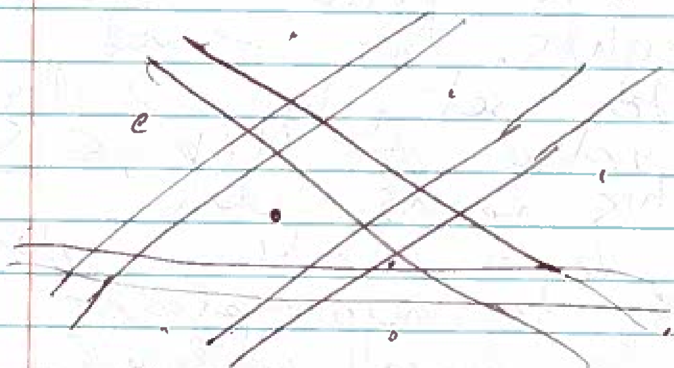
Theorem (*) and the commutation of elements of S w/ $\tilde{r}$ ~~[illegible]~~ the are a full presentation of the full category w/ objects $\{P_\lambda\}$.

Thm if deg on S is ~~double~~ usual and $\deg r(\lambda,\nu) = \#$ gaps w/ label θ crossed, relation is homogeneous.

Cor If λ, ν, χ all lie in one C_ε, then

$$r(\lambda,\nu)\, r(\nu,\chi) = r(\lambda,\chi).$$

This allows us to identify any two elements of the same C_ε (assuming I didn't choose my coset so degenerately that a C_ε has no integral points)

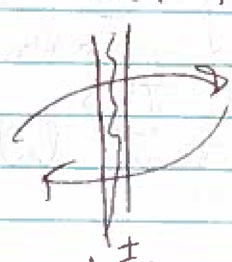


Pick a representative in each chamber. We can represent each by the sign vector.

~~The relations say that we can think of $r(\lambda,\nu)$ as a path~~

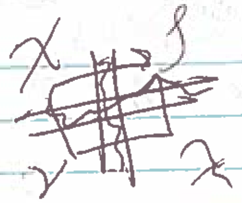
Again assuming genericity we can factor paths into elements of degree 1, crossing 1 hyperplane at a time.

What relations do these satisfy?



$$r(\lambda,\nu)\, r(\nu,\lambda) = h_i^{\eta}$$

$h_i^{\pm} = 0$

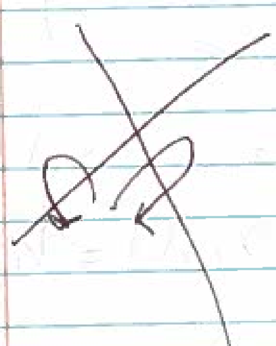


$$\tilde r(\xi, x)\,\tilde r(x,\nu)$$
$$= \tilde r(\xi,\nu)$$
$$= \tilde r(\xi,\lambda)\,\tilde r(\lambda,\nu)$$

Note that these relations are degree 2.

For our purposes, we want to assume that $Z(U)$ acts by a scalar, which is fixed by the choice of weights. This means we have to set to 0 the ~~nilpotent p~~ action of $U(\mathfrak{k}) \subseteq S$ to 0. This means when we have a linear relation between normal vectors to hyperplanes it holds w/ the same coefficients for $\hat{h}_i$'s.

This means that the action of S on any projective is generated by any normal vectors that form a basis.



So, the action of $\hat{S}$ can be written just using elements of degree 1. Thus, all the relations are quadratic! This will prove important later.

What about category $\mathcal{O}$?

Prop The simple weight module L_λ is in category $\mathcal{O}$ if the set C_ε is bounded above, and this is a full list of simples in category $\mathcal{O}$.

Theorem Every weight module M has a unique maximal quotient $M^{\mathcal{O}}$ that lies in category $\mathcal{O}$. This is the quotient of M by the image of every ~~P_χ~~ homomorphism ~~backerly~~ ~~although~~ from P_χ w/ $[\chi]$ unbounded.

This is left adjoint to inclusion of category $\mathcal{O}$ in weight modules
M weight, $N \in \mathcal{O}$

$$\mathrm{Hom}(M,N) \cong \mathrm{Hom}(M^{\mathcal{O}}, N).$$

Pf If M' "$= M/m$" is a quotient of M in category $\mathcal{O}$ then any map from P_χ w/ $[\chi]$ unbounded has trivial composition w/ the projection, since $\mathrm{Hom}(P_\chi, M') \cong (M')_\chi = 0$.

If we let M' be quotient by the sum of these images, then

$$M'_\chi = \mathrm{Hom}(P_\chi, M') = 0, \text{ so } M'$$

is in $\mathcal{O}$.

(Careful: why are weight spaces finite dimensional?)

For category $\mathcal{O}$, the object P_χ represents the functor $M \mapsto M_\chi$, so $P_\chi^{\mathcal{O}}$ is projective.

What is the relationship between

$$\mathrm{Hom}(P_\chi, P_\lambda) \quad ?$$
$$\mathrm{Hom}(P_\chi^{\mathcal{O}}, P_\chi^{\mathcal{O}})$$

$$\mathrm{Hom}(P_\chi, P_\lambda) \twoheadrightarrow \mathrm{Hom}(P_\chi, P_\lambda^{\mathcal{O}}) \cong \mathrm{Hom}(P_\chi^{\mathcal{O}}, P_\lambda^{\mathcal{O}})$$

Kernel is spanned by maps

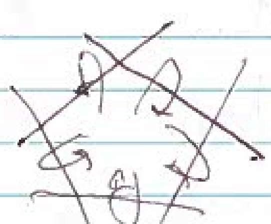
$$P_\chi \to P_\nu \to P_\lambda \quad \text{w/ } (\nu) \text{ unbounded.}$$

Choosing a rep of each chamber R C_ε, we can think about

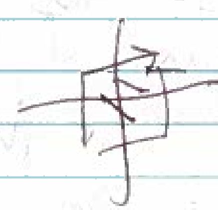
$$A = \mathrm{End}\left(\bigoplus P_\chi\right)$$

This has idempotents e_χ $\forall \chi \in R$
degree 1 paths $r_{\chi,\chi'}$ $\forall \chi, \chi' \in R$ sep by 1 hyperplane.

This has relations



satisfy linear rels of normal vectors



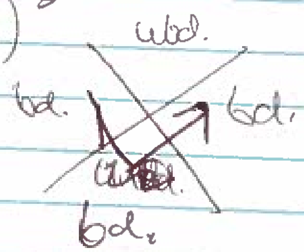
What are morphisms factoring through e_ν? The two sided ideal AeA.

Let $e = \sum_{[\nu] \text{ unbounded}} e_\nu$

$\boxed{\text{Then } \mathrm{End}\left(\bigoplus P_x^{\oplus}\right) \cong A/AeA.}$

We can still think of this as quadratic rels

*) we modify rels on wall-crossings by just deleting terms that go to unbounded chambers

*) 

$= 0.$

So, again category $\mathcal{O}$ is equivalent to modules over a graded quadratic algebra. This will prove important.