Weight modules

Now, let's study U-modules.

$U \supset D_0 \cong \mathbb{C}[h_i^+]$.

Assume that M is a f.d. U-module.

Lemma as a D_0-module, M has a decomposition

$$M \cong \bigoplus M_\chi \quad \text{for } \chi : D_0 \to \mathbb{C}$$

with $M_\chi = \{ v \in M \mid m_\chi^N \cdot v = 0 \text{ for } N \gg 0 \}$ and $m_\chi = \operatorname{Ker} \chi$, a maximal ideal.

Pf By JD, $h_i^+ = h_i^{ss} + h_i^n$. The h_i^{ss} are s.s. and commute, so are simultaneously diagonalizable. For each basis vector, we have some χ s.t. $h_i^{ss} \cdot v = \chi(h_i^+) \cdot v$. M_χ is the span of such vectors, since ~~the~~ $h_i^+ - \chi(h_i^+)$ generate this ideal, and

$$h_i^+ - \chi(h_i^+) \cdot v = (h_i^n + h_i^{ss} - \chi(h_i^+)) \cdot v = h_i^n \cdot v.$$

which is nilpotent by def.

Prop On U-mod f.d., h_i^n ~~defines~~ defines an element of the Bernstein center, that is, by endomorphisms that commute with all U-module homomo

Pf This works because the

adjoint action of ~~ad~~ $\mathfrak{D}_0$ on $\mathcal{U}$ is semi-simple. $\mathcal{U}$ is spanned by vectors s.t. $[h, a] = \xi(h)\, a$ for some $\xi : \mathfrak{D}_0 \to \mathbb{C}$. h_i^n commutes w/ such an a because on a vector $v \in M_\chi$, we have

$$h_i^n \cdot av = (h_i^+ - h_i^{ss}) \cdot av = h_i^+ av - \chi + \xi(\mathrm{ad}\, h_i^+) av$$
$$= a h_i^+ v + \xi(h_i^+) av - \chi + \xi(h_i^+) av$$
$$= a h_i^+ v - \chi(h_i^+) av = a \cdot h_i^n v$$

Key point: $av \in M_{\chi+\xi}$, so $[h_i^{ss}, a] = \xi(h) a$.

This shows h_i^n is an endomorphism. It is natural since any homomorphism sends $M_\chi \to N_\chi$

and on these spaces $h_i^n = h_i^+ - \chi(h_i^+)$.

Lemma: on any simple, $S = \mathbb{C}[h_i^n, \ldots, h_i^n]$ acts trivially.

Pf: The vectors killed by all h_i^n are a non-zero submodule, and thus everything.

Note: I have only used the weight decomposition, not any actual finite dimensionality.

Def If M is a finitely generated $U(\mathfrak{g})$-module, we call M a **weight module** if it is D_0-locally finite, i.e. for any f.d. subspace $N \subset M$, $D_0 \cdot N$ is also finite dim.

Def The **class** of a M weight module is the length of the shortest filtration such that
$$h_i^n \cdot M^{(n)} \subseteq M^{(n+1)}.$$

(You can take socle filtration)
$$M^{(0)} = M, \quad M^{(n+1)} = \sum h_i^n M^{(n)}$$

Lemma Assume M is a weight module. If N is a f.d. generating subspace, the class of M is equal to that of $D_0 \cdot N$, and the dimension of $M_\chi \leq \dim D_0 \cdot N$.

Pf. M_χ is spanned by $r_{\chi - \lambda} \cdot n$ for $n \in (D_0 \cdot N)_\lambda$. This gives dimension bound. The layers of the socle filtration are the submodules gen by the layers of the socle filtration of $D_0 \cdot N$.

So, why do this? The category of weight modules is better behaved.

In particular, we have the exact functor

$$M \mapsto M_\chi = \{v \mid m_\chi^N \cdot v = 0 \quad N \gg 0\}.$$

It's maybe better to start with

$$M_\chi^{(n)} = \{v \in M \mid m_\chi^n \cdot v = 0\}.$$

Since this is defined by vanishing of a left ideal $\mathcal{U} \cdot m_\chi^n$ this functor is represented by a cyclic module

$$\mathrm{Hom}(\mathcal{U}/I\mathcal{U}, M) \cong \{v \in M \mid Iv = 0\}.$$

$$\text{Let } P_\chi^{(n)} = \mathcal{U}/\mathcal{U} \cdot m_\chi^n.$$

<u>Prop</u> $\mathrm{Hom}(P_\chi^{(n)}, M) \cong M_\chi^{(n)}$

$$\begin{pmatrix} w = \bar{1} \longmapsto v \\ \bar{u} \longmapsto uv. \end{pmatrix} \longleftrightarrow v \in M_\chi^{(n)}$$

Note $M_\chi^{(1)} = \{v \in M_\chi \mid h_i^u \cdot v = 0\}$.

<u>Thm</u> If L is a simple module, and $L_\chi \neq 0$, then L is a quotient of $P_\chi^{(1)}$.

<u>Pf</u> We have an induced map $P_\chi^{(1)} \to L$ which is not 0, $w \longmapsto v \in L_\chi \setminus \{0\}$ and thus surjective.

What does $P_\chi^{(1)}$ look like?

<u>Prop</u> $\{m^\lambda w\}_{\lambda \in \mathbb{Z}_{\geq 0}^{\Phi^+}}$ is a basis for $P_\chi^{(1)}$.

Pf $P_\chi^{(1)} \cong \mathcal{U} \otimes_{D_0} \mathbb{C}_\chi$, and m^λ are a basis for $\mathcal{U}$ as a right D_0-module.

Cor $\dim(P_x^{(1)})_\nu = \begin{cases} 1 & x-\nu \in t\hat{\mathbb{Z}} \\ 0 & \text{otherwise.} \end{cases}$

Cor A submodule $N \subset P_x^{(1)}$ is proper iff $N_x = 0$.

Pf If $\dim N_x > 0$, then $N_x = (P_x^{(1)})_x = \mathbb{C}\cdot w$ so $N = P_x^{(1)}$.

Cor The module $P_x^{(1)}$ has a unique simple quotient.

Pf If N, N' are proper submodules, then $N + N'$ is proper as well. ~~thus the sum of all proper submodules~~ After all $(N+N')_x = N_x + N'_x = 0$. Thus If $0 \to K \to P_x^{(1)} \to L \to 0$ is a SES w/ simple quotient, and N is any proper submodule, then $N + K = K$ or $N + K = P_x^{(1)}$ since L is simple, but the latter is impossible. Thus $N \leq K$, which shows there is no other maximal submodule w/ simple quotient.

Let L_x be the unique simple quotient.

<u>Prop.</u> TFAE 0) $P_x^{(1)} \cong P_\nu^{(1)}$

1) $L_x \cong L_\nu$

2) $\dim(L_x)_\nu = 1$

3) $\dim(L_x)_\nu > 0$

4) $r_{x-\nu} r_{\nu-x} w \neq 0 \in P_x^{(1)}$

1) => 2) $\dim(L_\nu)_\nu = 1$, since $(L_\nu)_\nu = \mathbb{C}\cdot$

2) => 3) obvious

3) => 4) if $\dim(L_x)_\nu > 0$, then $\exists\, a \in U_{\nu-x}$ s.t. ~~$a w \neq 0$ since $a = f r_{\nu-x}$ for $f \in D_0$, $r_{\nu-x} w \neq 0$~~ aw generates $P_x^{(1)}$. For some $f \in D_0$ $a = f r_{\nu-x}$, for $r_{\nu-x} w$ generates. Thus $w = b r_{\nu-x} w$ w/ $b \in U_{x-\nu}$. Since $b = g r_{x-\nu}$, we have

$$w = g r_{x-\nu} r_{\nu-x} w \Rightarrow r_{x-\nu} r_{\nu-x} w \neq 0$$

aw for $a \in \mathbb{C} \setminus \{0\}$

4) => 0). $r_{x-\nu} r_{\nu-x} w = aw$ so $r_{x-\nu} r_{\nu-x} w$ generates $P_x^{(1)}$, so $r_{\nu-x} w$ generates; this gives a map $P_\nu^{(1)} \to P_x^{(1)}$. Similarly we have an induced map $P_x^{(1)} \to P_\nu^{(1)}$. The composition of these maps is mult. by a, and thus they are isomorphisms

0) => 1) The unique quotients of isomorphic varieties are isomorphic.

We can thus define an equivalence relation $x \sim \nu$ if a condition above holds.

<u>Prop</u> Simple weight modules over U are indexed by EC in this relation.

Let $\cancel{x_i = t_i}$ $x_i - \nu_i = h_i^{\pm}(x) - h_i^{\pm}(\nu)$

$$r_{x-\nu} r_{\nu-x} = \prod_{i=1}^{n} [h_i]^{x_i - \nu_i}$$

$$\|$$

$$(h_i^- - 1) \cdots (h_i^- - x_i + \nu_i) \quad x_i \geq \nu_i$$

or

$$(h_i^+ + 1) \cdots (h_i^+ - x_i + \nu_i) \quad x_i \leq \nu_i$$

Thus $x \sim \nu$ iff $x - \nu \in t^*_{\mathbb{Z}}$ and the polynomial above is non-zero at x. $\iff$ for all i, $x_i \geq \nu_i$ and

$$h_i^-(x) \notin \{1, \ldots, x_i - \nu_i\}$$
$$h_i^+(x) \notin \{0, \ldots, x_i - \nu_i - 1\}$$
$$h_i^-(\nu) \notin \{\nu_i - x_i + 1, \ldots, 0\}$$
$$h_i^+(\nu) \notin \{\nu_i - x_i, \ldots, -1\}.$$

or

$x_i \leq \nu_i$ and

$$h_i^+(x) \notin \{x_i - \nu_i, \ldots, -1\}$$
$$h_i^-(x) \notin \{x_i - \nu_i + 1, \ldots, 0\}$$
$$h_i^+(\nu) \notin \{0, \ldots, \nu_i - x_i - 1\}$$
$$h_i^-(\nu) \in \{1, \ldots, \nu_i - x_i\}.$$

$\iff$ we don't have $h_i^-(x) \in \mathbb{Z}_{>0}$ and $h_i^+(\nu) \in \mathbb{Z}_{<0}$ nor $h_i^+(x) \in \mathbb{Z}_{<0}$ and $h_i^-(\nu) \in \mathbb{Z}_{>0}$.

Thm $x \sim \nu$ iff $x - \nu \in t^*_{\mathbb{Z}}$ and there is no i such that

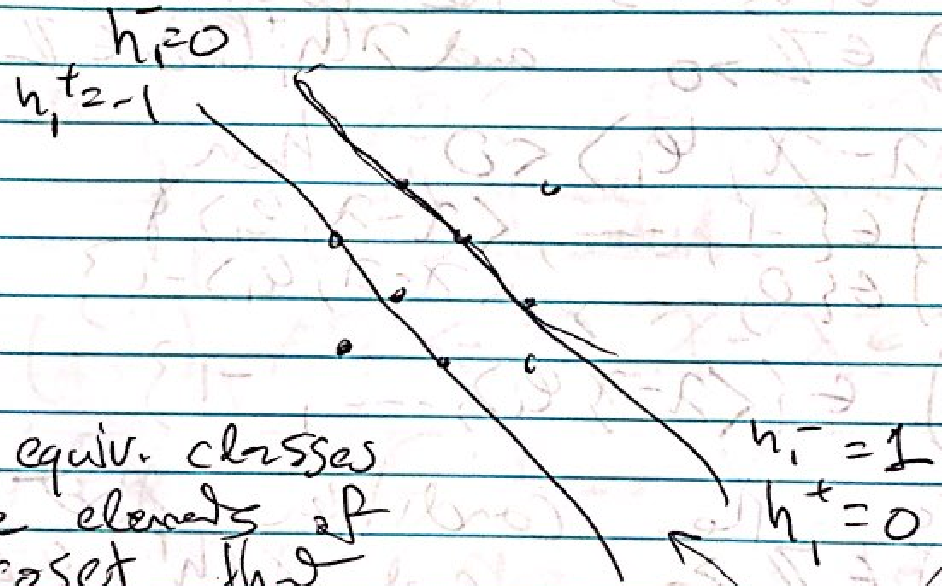
1) $h_i^-(x) \in \mathbb{Z}_{>0}$ and $h_i^+(\nu) \in \mathbb{Z}_{<0}$

or 2) $h_i^-(\nu) \in \mathbb{Z}_{>0}$ and $h_i^+(x) \in \mathbb{Z}_{<0}$

More geometrically, we can restrict to a J' coset of $t^*_{\mathbb{Z}}$.

On this coset either $h_i^\pm$ is integral on all elements or none. If it is integral on none, it makes no contribution, and we can ignore it.

If it is integral everywhere we can draw in the zero sets of $h_i^\pm$.



$h_i^- \in \mathbb{Z}_{\geq 0}$

no integral pts here

$h_i^+ \in \mathbb{Z}_{<0}$

Thus, equiv. classes are the elements of the coset that lie in a polytope defined by the equations

or $h_i^+ \geq 0$ $(h_i^- \geq 1)$
$h_i^- \leq 0$ $(h_i^+ \leq -1)$,

for each "relevant" i.

Let C_ε for $\varepsilon \in \{+,-\}^n$

$= \{\lambda \in t^*_{\mathbb{Z}} + \nu \mid \varepsilon(h_i^+(\lambda) + 1/2) = \varepsilon(h_i^-(\lambda) - 1/2) > 0\}$.

How do you think about these arrangements? Each coset of $\mathfrak{t}^*_{\mathbb{Z}}$ lies in one of $\mathfrak{t}^*$. This requires choosing an element of $\mathfrak{k}^{*} \cong \mathbb{C}^n / \mathfrak{t}^*$

The intersection ~~of~~ vanishing sets of $h_i^{\pm}$ define a doubled hyperplane arrangement in our coset, which is given by taking the hyperplane Nick drew and pushing the hyperplanes off the origin (and doubling them)

Then, for a given coset of $\mathfrak{t}^*_{\mathbb{Z}}$, ~~thou~~ we must throw out the walls that don't have integral value.

For example, let $K = (\mathbb{C}^*)^n \cap SL(n)$. In this case, $T \cong \mathbb{C}^*$ via taking det.

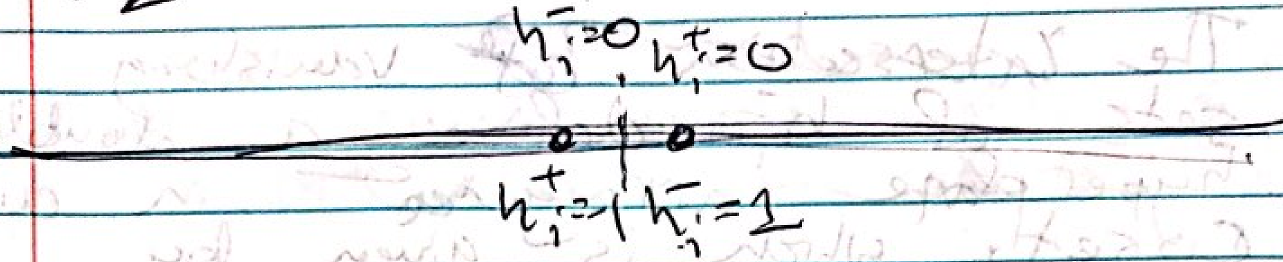
$\mathfrak{t}^* \subset \mathbb{C}^n = \{(x, \dots, x) \mid x \in \mathbb{C}\}$

$\mathfrak{k}^* \cong \{(-a_1, \dots, -a_n) \mid \sum a_i = 0\}$

Note that for a given coset, the functions h_i^+ are all just translates of the coordinate x by $-a_i$.

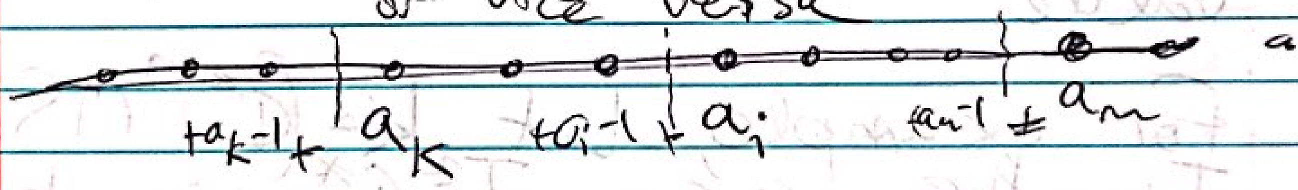
~~Note~~ We don't have a canonical coordinate x; this is only an affine space. What's canonical is the relations $h_i^+ - h_k^+ = a_i + a_k$.

For any h_i^{I} the set of integral values is itself a $\mathbb{Z}_{\geq} \cup \mathbb{Z}_{\leq}$ coset



~~4A~~ ~~2x~~ We have $h_i^+(x) = x + a_i$
$h_i^-(x) = x + a_i + 1$

$x \sim y$ if $x - y \in \mathbb{Z}$ and there is no a_i s.t. $x + a_i \in \mathbb{Z}_{<0}$ and $y + a_i \in \mathbb{Z}_{\geq 0}$ or vice versa

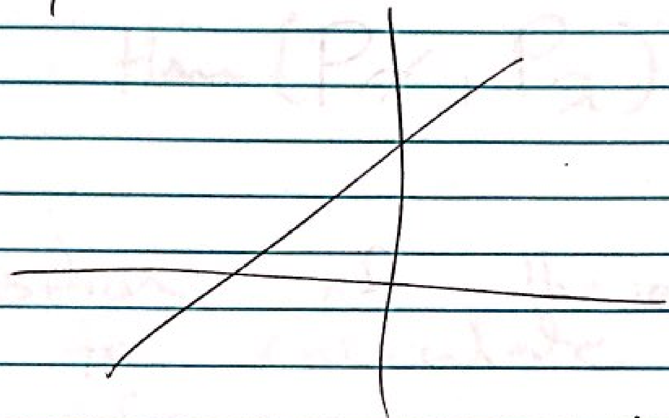


Note: # of fid. simples for coset $= n = $ # of $\mathbb{Z}$ cosets a_i the

dual situation, $K = \mathbb{C}^{\times} \cdot T$
T = maximal torus of $PGL(n)$

Coset of $t_V^{\times}$ = fix sum $\sum a_i = k$.
Coset of $t_{\mathbb{Z}}^{\times}$ - fix sum and cosets of coordinates in $\mathbb{C}/\mathbb{Z}$ (which must sum to k mod $\mathbb{Z}$).

Hyper planes define $(n-1)$-simplex



For a given coset, behavior depends on # on integral coordina

$k \notin \mathbb{Z}$: can have $0, 1, \ldots, n-1$ integral coordinates (n impossible)

$k \in \mathbb{Z}$: can have $0, 1, \ldots, n-2, n$ integral coordinates ($n-1$ impossible.

Only have f.d. reps if $k \in \mathbb{Z}$ and only one then.

$\text{Gr dim}(L_x)_\lambda = \begin{cases} 1 & x \sim \lambda \\ 0 & \text{otherwise} \end{cases}$

We don't want to just assume that S acts trivially. What about when we have some nilpotent part?

As I said before, the weight space functor $M \mapsto M_\chi$ is represented by

$$P_\chi = \varprojlim P_\chi^{(N)} := U/U m_\chi^N$$
$$\cong U \otimes_{D_0} D_0/m_\chi^N$$

This not a weight module but a topological one! Another way to think about this space is that we put a topology on U with $U m_\chi^N$ being a basis of neighborhoods of identity. We can always complete a topological group with respect to any topology (respecting the group structure)

(A Cauchy sequence $\{a_n\}$ is one s.t. for any neighborhood of the identity U $\exists N$ s.t. for $n,m > N$, $a_n - a_m \in U$.)

A topological weight vector in this ~~cliq~~ module is a vector v s.t. for any nbhd $W \ni 0$, $\exists N$ s.t. $m_\chi^N \cdot v \in W$. $\hat{S} = S$ completed at 0 acts on topological weight spaces by completeness

If ~~one~~ V is image of 1, then V is a TWV of weight χ (essentially by def) and

$$\hat{S} \cdot r_{\lambda-\chi} \cdot V = (P_\chi)_\lambda .$$

Theorem Every weight module is a quotient of a sum of $\oplus P_{\chi_i}^{\oplus a_i}$.

Understanding projectives is very important because the kernel of the map $\oplus P_\chi$ $\oplus P_{\chi_i}^{\oplus a_i} \to M$ itself is the image of a map from a sum of projectives itself