WARTHOG 2017

Hypertoric varieties and their lagrangians

$\mathcal{V} = (V, \eta, \xi)$ polarized arrangement

- $V \subset \mathbb{R}^I$ linear subspace
- $\eta \in \mathbb{R}^I / V \rightsquigarrow V_\eta := V + \eta \subset \mathbb{R}^I$ affine subspace

$\uparrow \xi$

$\Delta = \Delta_{+++}$
$= V_\eta \cap \mathbb{R}^I_{\geq 0}$

$H_i : V_\eta \cap i^{th}$ coord hyperplane

- $\xi \in V^* = (\mathbb{R}^I)^* / V^\perp$ linear function on V
  linear function up to constant on V^*

- $1 \longrightarrow K \longrightarrow (\mathbb{C}^*)^I \longrightarrow T \longrightarrow 1$

$$V = \mathfrak{t}^*_{\mathbb{R}}$$
$$\mathbb{R}^I / V = \mathfrak{k}^*_{\mathbb{R}}$$

$$\eta \in \mathfrak{k}^*_{\mathbb{Z}} \cong \mathrm{Hom}(K, \mathbb{C}^*)$$
$$\xi \in V^*_{\mathbb{Z}} \cong \mathfrak{t}_{\mathbb{Z}} \cong \mathrm{Hom}(\mathbb{C}^*, T).$$

Let's start by defining the toric variety $X(\nu)$.

lattice points in $k\Delta$

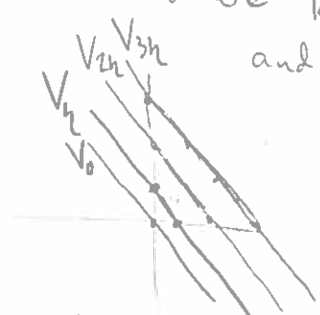
Def: $$R(\nu) = \bigoplus_{k\geq 0} \mathbb{C}\{V_{k\eta} \cap \mathbb{N}^I\} \subset \bigoplus_{k\geq 0} \mathbb{C}\{\mathbb{N}^I\}$$

$$\left(\mathbb{C}[x_i]_{i\in I} \otimes \mathbb{C}[t]\right)^k, \qquad \mathbb{C}[x_i]_{i\in I} \otimes \mathbb{C}[t]$$

where $k \cdot x_i = k_i x_i$ and $k \cdot t = \eta(k)^{-1} t$

This is because $V_{k\eta}$ is the η^k-eigenspace

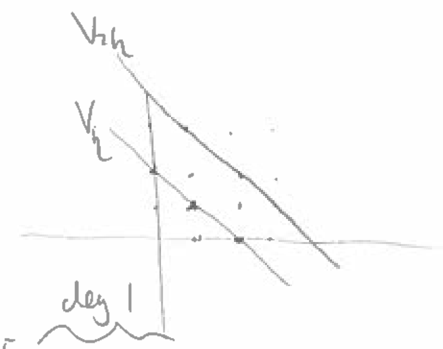
Examples: (1) $I = \{1,2\}$



$$R(\nu) = \mathbb{C}[\underbrace{y_1, y_2}_{\deg 1}] \hookrightarrow \mathbb{C}[x_1, x_2, t]$$

$$y_i \longmapsto x_i t$$

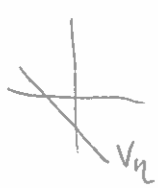
(2) $I = \{1, 2\}$



$$R(\nu) = \mathbb{C}[\underbrace{a,b,c}_{\deg 1}] / \langle b^2 - ac\rangle \hookrightarrow \mathbb{C}[x_1, x_2, t]$$

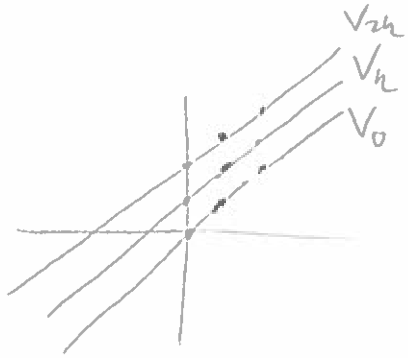
$$a \longmapsto x_1^2 t$$
$$b \longmapsto x_1 x_2 t$$
$$c \longmapsto x_2^2 t$$

(3) $I = \{1,2\}$



$R(\nu) = \mathbb{C}$

4 $I = \{1, 2\}$



$$R(\nu) : \mathbb{C}[\underset{\deg 0}{x}, \underset{\deg 1}{y}] \hookrightarrow \mathbb{C}[x_1, x_2, t]$$

$$x \longmapsto x_1 x_2$$

$$y \longmapsto x_2 t$$

Def: $X(\nu) := \operatorname{Proj} R(\nu)$ "toric variety"

Crash Course on Proj: $R = \bigoplus_{k \geq 0} R_k$ $\quad \mathbb{C}^\times \curvearrowright \operatorname{Spec} R$

$$R_0 \hookrightarrow R \twoheadrightarrow R_0$$

$$\operatorname{Spec} R_0 \twoheadleftarrow \operatorname{Spec} R \hookleftarrow \operatorname{Spec} R_0 .$$

$$\operatorname{Proj} R := (\operatorname{Spec} R \setminus \operatorname{Spec} R_0)/\mathbb{C}^\times$$

$$\downarrow$$

$$\operatorname{Spec} R_0$$

Let's look at our examples

1. $R = \mathbb{C}[\underbrace{y_1, y_2}_{\deg 1}]$ $\quad$ $\operatorname{Spec} R = \mathbb{C}^2$
   $\operatorname{Spec} R_0 = pt$

$$\begin{array}{ccl} \operatorname{Proj} R & = & (\mathbb{C}^2 \setminus pt)/\mathbb{C}^* = \mathbb{CP}^1 \\ \downarrow & & \\ \operatorname{Spec} R_0 & = & pt \end{array}$$

More generally, Δ std d-simplex $\rightsquigarrow$ poly ring on $d+1$ gens in degree 1 $\rightsquigarrow \mathbb{CP}^d$

2. $R = \mathbb{C}[a,b,c]/\langle b^2 - ac \rangle$

$\operatorname{Spec} R \subset \mathbb{C}^3$
$\operatorname{Proj} R \subset \mathbb{CP}^2$ quadric
$\cong$
$\mathbb{CP}^1$ in Veronese embedding

More generally: scaling Δ doesn't change the isomorphism type of $X(\Delta)$

(We'll see this more explicitly in a minute)

3. $R = \mathbb{C} = R_0$
   $\operatorname{Proj} R = \emptyset$ $\quad$ More generally, if $\Delta = \emptyset$, $X(\Delta) = \emptyset$

4 $R = \mathbb{C}[x, y]$ $\quad R_0 = \mathbb{C}[x]$

$$\underset{\deg 0}{x}, \quad \underset{\deg 1}{y}$$

$$\operatorname{Proj} R = \mathbb{C}^2 \setminus (\mathbb{C} \times \{0\}) \big/ \mathbb{C}^*$$

$$= (\mathbb{C} \times \mathbb{C}^*) / \mathbb{C}^*$$

$$\cong \mathbb{C}$$

More generally: $\Delta \cong \mathbb{R}^d_{\geq 0}$

$\rightsquigarrow R \cong \mathbb{C}[x_1, \ldots, x_d, y]$

$\rightsquigarrow X \cong \mathbb{C}^d$

What happens when we have a graded homomorphism?

$$\begin{tikzcd}
R \arrow[r, "\varphi"] & S \\
\operatorname{Spec} R \arrow[d, phantom, "\cup"] & \operatorname{Spec} S \arrow[l, "\varphi^*"'] \arrow[d, phantom, "\cup"] \\
\operatorname{Spec} R_0 & \operatorname{Spec} S_0 \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{Proj} R & \operatorname{Proj} S \arrow[l, dashed] \arrow[d, phantom, "\cup"] \\
& \left\{ z \in \operatorname{Proj} S \;\middle|\; \begin{array}{l} \text{if } \tilde{z} \in \operatorname{Spec} S \text{ is a lift,} \\ \varphi^* \tilde{z} \notin \operatorname{Spec} R_0 \end{array} \right\} \arrow[ul]
\end{tikzcd}$$

$$\|$$

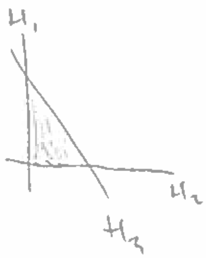
$$\{ z \mid \exists\, f \in R_{>0} \text{ st } \varphi(f)(\tilde{z}) \neq 0 \}$$

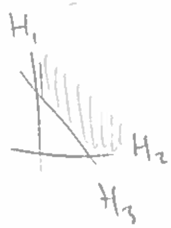
We have $R(\mathcal{V}) \hookrightarrow \mathbb{C}[x_i]_{i\in I} \otimes \mathbb{C}[t]$

$$\begin{tikzcd} X(\mathcal{V}) & \mathbb{C}^I \arrow[l, dashed] \\ & (\mathbb{C}^I)^{st} \arrow[u, phantom, "\cup"] \arrow[ul] \end{tikzcd}$$

The locus on which this map is defined is called the "stable locus"

Prop: $(\mathbb{C}^I)^{st} = \left\{ z \;\middle|\; \Delta \cap \bigcap_{z_i=0} H_i \neq \emptyset \right\}$

Ex:  $(\mathbb{C}^3)^{st} = \mathbb{C}^3 \setminus \{0\}$

 $(\mathbb{C}^3)^{st} = \mathbb{C}^3 \setminus \{0\} \times \mathbb{C} \times \{0\}$.

$\Delta = \emptyset \Rightarrow (\mathbb{C}^I)^{st} = \emptyset$

"Pf"
$$\begin{array}{ccc} R_k & \hookrightarrow & \mathbb{C}[x_i]_{i\in I} \cdot t^k \\ \shortparallel & & \\ \mathbb{C}\{\Delta^{k\mathcal{V}} \cap \mathbb{N}^I\} & & \\ \ni & & \ni \\ e & \longmapsto & x^e t^k \end{array}$$

$x^e t^k(\tilde{z}) \neq 0 \iff e_i = 0$ whenever $z_i = 0$

$\iff e \in \bigcap_{z_i=0} H_i$.

By our integrality assumptions, if the face is nonempty for some k it's nonempty

Fact (GIT basics): $X(\nu) = (\mathbb{C}^I)^{st}/K$

ie the map from $(\mathbb{C}^I)^{st}$ is surjective, and two points go to the same place iff they lie on the same K-orbit.

Have $(\mathbb{C}^\times)^I \curvearrowright (\mathbb{C}^I)^{st}$

$$\begin{array}{ccc} (\mathbb{C}^\times)^I/K & \curvearrowright & (\mathbb{C}^I)^{st}/K \\ \wr\wr & & \wr\wr \\ T & & X(\nu) \end{array}$$

$\xi \in \mathrm{Hom}(\mathbb{C}^\times, T) \rightsquigarrow \mathbb{C}^\times \curvearrowright X(\nu)$.

Q: Does $\lim_{t\to\infty} t\cdot x$ exist $\forall x \in X(\nu)$?

Nontrivial question: If $X(\nu) \cong \mathbb{C}P^1$, then sure

If $X(\nu) \cong \mathbb{C}$, then it depends on ξ.

Prop: Assume $\Delta \neq \emptyset$ (so $X(\nu) \neq \emptyset$).

$\lim_{t\to\infty} t\cdot x$ exists $\forall x \in X(\nu) \iff \Delta$ is ξ-bounded.

Pf of $\Rightarrow$: Let $x = [(1, \ldots, 1)] \in (\mathbb{C}^I)^{st}/K = X(\nu)$.

Choose $\tilde{\xi} \in \mathrm{Hom}(\mathbb{C}^\times, (\mathbb{C}^\times)^I) \cong \mathbb{Z}^I$

lifting $\xi \in \mathrm{Hom}(\mathbb{C}^\times, T)$.

$$\text{Then } \lim_{t\to\infty} \xi(t)\cdot x = \lim_{t\to\infty} [\xi(t)(1, \ldots, 1)]$$

$$= \lim_{t\to\infty} [(t^{\tilde{\xi}_i})_{i\in I}]$$

If this limit exists, then $\exists\ \delta \in \underset{\underset{\textstyle \mathrm{Hom}(\mathbb{C}^\times, (\mathbb{C}^\times)^I) \cong \mathbb{Z}^I}{\cap}}{\overset{V_{\mathbb{Z}}^\perp = }{\mathrm{Hom}(\mathbb{C}^\times, K)}}$

st $\tilde{\xi}_i - \delta_i \le 0 \quad \forall i$.

$\Rightarrow \delta - \xi \in \mathbb{R}^I_{\ge 0}$

$\Rightarrow \Delta^! = (V^\perp - \xi) \cap \mathbb{R}^I_{\ge 0} \neq \emptyset$

$\Rightarrow \Delta^!$ feasible

$\Rightarrow \Delta$ bounded.

I'll leave it to you to convince yourselves that all of these steps are reversible.

We've defined a variety with some nice connections to the combinatorics of $\Delta = \Delta_{+\cdots+}$. How about the other Δ_α?

Def: $R_\alpha(\nu) = \left(\mathbb{C}[x_i]_{\alpha_i=+1} \otimes \mathbb{C}[y_j]_{\alpha_j=-1} \otimes \mathbb{C}[t] \right)^K$

where
$$k \cdot x_i = k_i x_i$$
$$k \cdot y_j = k_j^{-1} y_j$$
$$k \cdot t = \nu(k)^{-1} t$$

$$X_\alpha(\nu) = \operatorname{Proj} R_\alpha(\nu)$$

Prop: • $X_\alpha(\nu) \cong (\mathbb{C}^I_\alpha)^{st}/K$, where $(\mathbb{C}^I_\alpha)^{st} = \{ z \mid \Delta_\alpha \cap \bigcap_{z_i=0} H_i \neq \emptyset \}$

• Assuming $\Delta_\alpha \neq \emptyset$,
$\lim_{t\to\infty} \xi(t)\, x$ exists $\forall\, x \in X_\alpha(\nu) \iff \Delta_\alpha$ ξ-bounded.

Okay, that covers what I wanted to say about toric varieties. Now a brief word about hypertoric varieties!

Def: $S(\mathcal{V}) := \left(\mathbb{C}[x_i, y_i]_{i\in I} \otimes \mathbb{C}[t]\right)^k$.

$L(\mathcal{V}) := \operatorname{Proj} S(\mathcal{V})$. "Lawrence toric variety"

Notes: ① $\forall \alpha$, $S(\mathcal{V}) \twoheadrightarrow R_\alpha(\mathcal{V})$

$L(\mathcal{V}) \hookleftarrow X_\alpha(\mathcal{V})$

② $S(\mathcal{V}) \hookrightarrow \mathbb{C}[x,y,t]$

$$\begin{array}{ccc} L(\mathcal{V}) & \dashleftarrow & \operatorname{Proj} \mathbb{C}[x,y,t] \\ & & \parallel \\ & & \mathbb{C}^I \times \mathbb{C}^I \\ & \nwarrow & \cup \\ & & (\mathbb{C}^I \times \mathbb{C}^I)^{st} \end{array}$$

Prop: $(\mathbb{C}^I \times \mathbb{C}^I)^{st} = \left\{ (z,w) \;\middle|\; V_h \cap \bigcap_{z_i=0} H_i \cap \bigcap_{w_i=0} H_i^+ = \emptyset \right\}$,

and $L(\mathcal{V}) = (\mathbb{C}^I \times \mathbb{C}^I)^{st}/K$

(Note that, if $w_i = 0\ \forall i$, this gives us our old prop.)

③ $L(\mathcal{V}) = \operatorname{Proj} S(\mathcal{V})$

resolution of singularities $\downarrow$ $\qquad\downarrow$

$L(\mathcal{V})_0 := \operatorname{Spec} S(\mathcal{V})_0$

$$\begin{aligned} S(\mathcal{V})_0 &= \mathbb{C}[x,y]^k \\ &\cong \operatorname{gr} \mathbb{C}[x,\partial]^k \\ &= \operatorname{gr} U(\mathcal{V}) \end{aligned}$$

hypertoric enveloping algebra

We have

$$\begin{array}{ccccc} \operatorname{Sym} \mathbb{C}^n & = & \mathbb{C}[x_i \partial_i]_{i\in I} & \subset & U(\mathcal{V}) \\ \cup & & & & \shortparallel \\ \operatorname{Sym} \mathfrak{k} & & \cong & & Z(U(\mathcal{V})) \end{array}$$

For any $\lambda \in \mathfrak{k}^*$, we defined $U_\lambda(\mathcal{V}) := U(\mathcal{V}) \otimes_{\mathrm{Sym}\,\mathfrak{k}} \mathbb{C}_\lambda$. $\mathfrak{k}$ acts by λ

Taking gr: $\mathrm{Sym}\,\mathbb{C}^n = \mathbb{C}[x_i y_i]_{i \in I} \subset S(\mathcal{V})_0$

$$\begin{array}{c} \mathrm{Sym}\,\mathbb{C}^n \\ \cup \\ \mathrm{Sym}\,\mathfrak{k} \end{array}$$

Prop: $\mathrm{gr}\, U_\lambda(\mathcal{V}) \cong S(\mathcal{V})_0 \otimes_{\mathrm{Sym}\,\mathfrak{k}} \mathbb{C}_0$ (no matter what λ is).

Def: $Y(\mathcal{V}) := \mathrm{Proj}\left(S(\mathcal{V}) \otimes_{\mathrm{Sym}\,\mathfrak{k}} \mathbb{C}_0\right)$ "hypertoric variety"

$\downarrow$ resolution

$Y(\mathcal{V})_0 := \mathrm{Spec}\left(S(\mathcal{V})_0 \otimes_{\mathrm{Sym}\,\mathfrak{k}} \mathbb{C}_0\right) = \mathrm{Spec}\left(\mathrm{gr}\, U_\lambda(\mathcal{V})\right)$

Note that $S(\mathcal{V}) \twoheadrightarrow S(\mathcal{V}) \otimes \mathbb{C}_0 \twoheadrightarrow R_\alpha(\mathcal{V})$,

so $L(\mathcal{V}) \hookrightarrow Y(\mathcal{V}) \hookleftarrow X_\alpha(\mathcal{V})$

$\bigcup_\alpha X_\alpha(\mathcal{V}) \subset Y(\mathcal{V})$ "extended core"

$\bigcup_{\alpha\ \xi\text{-bounded}} X_\alpha(\mathcal{V}) \subset Y(\mathcal{V})$ "relative core" we've stated one inclusion already

Prop: relative core $= \{ y \in Y(\mathcal{V}) \mid \lim_{t \to \infty} \xi(t) \cdot y \text{ exists} \}$

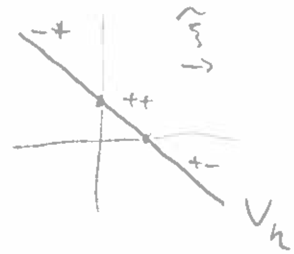
We need an example!

Ex: $I = \{1,2\}$

$V = \{z \mid z_1 + z_2 = 0\}$

$\tilde{\eta} = (1,0) \in \mathbb{R}^2$

$\tilde{\xi} = (1,0) \in (\mathbb{R}^2)^*$



Before we do any difficult calculations:

Extended core $\cong$ $\mathbb{C} \cup \mathbb{CP}^1 \cup \mathbb{C}$

Rel core $\cong$ $\mathbb{C} \cup \mathbb{CP}^1$

$$1 \longrightarrow \underset{\mathbb{C}^\times_\Delta}{\overset{\shortparallel}{K}} \longrightarrow (\mathbb{C}^\times)^2 \longrightarrow T \longrightarrow 1 \qquad \eta : \mathbb{C}^\times_\Delta \xrightarrow{\text{id}} \mathbb{C}^\times$$

$$L(\nu) = (\mathbb{C}^2 \times \mathbb{C}^2)^{st} / K$$

$$= \left\{ \begin{pmatrix} z_1 & z_2 \\ w_1 & w_2 \end{pmatrix} \;\middle|\; z_1 \text{ and } z_2 \text{ not both } 0 \right\} \Big/ \mathbb{C}^\times_\Delta$$

Acting on z with wt -1
w with wt 1

$$Y(\nu) = \left\{ \begin{pmatrix} z_1 & z_2 \\ w_1 & w_2 \end{pmatrix} \;\middle|\; \begin{array}{l} z_1, z_2 \text{ not both } 0 \\ z_1 w_1 + z_2 w_2 = 0 \end{array} \right\} \Big/ \mathbb{C}^\times$$

$$Y(\nu) \longrightarrow \mathbb{CP}^1 \qquad \text{Fiber is a line.}$$
$$[\,{}^{z_1\ z_2}_{w_1\ w_2}\,] \longmapsto [z_1, z_2] \qquad \text{Prop. } Y(\nu) \cong T^*\mathbb{CP}^1$$

What about the relative core?

$$\mathbb{C}^\times \circlearrowright \mathbb{CP}^1 \rightsquigarrow \mathbb{C}^\times \circlearrowright T^*\mathbb{CP}^1$$

$$t \cdot [z_1, z_2] = [tz_1 : z_2] \qquad t\cdot\begin{bmatrix} z_1 & z_2 \\ w_1 & w_2 \end{bmatrix} = \begin{bmatrix} tz_1 & z_2 \\ t^{-1}w_1 & w_2 \end{bmatrix}$$

Who has a limit as $t \to \infty$?

$\begin{bmatrix} z_1 & z_2 \\ 0 & 0 \end{bmatrix}$ $\mathbb{CP}^1$

$\begin{bmatrix} 1 & 0 \\ 0 & w_2 \end{bmatrix}$ $\mathbb{C}$