WARTHOG 2016
Exercises on Preliminary Reading

1. (a) Write down the standard basis of crossingless planar tangles for TL_3. Show that as an algebra, TL_3 is generated by the tangles $\mathbf{B}_1$ and $\mathbf{B}_2$ below. Using this, write down the multiplication table for TL_3 with respect to this basis.

   (b) A crossingless planar n-tangle is said to have k *turnbacks* if there are k arcs of the tangle which both start and end on the lower boundary. Show that TL_n is filtered by ideals: $I_{\lfloor \frac{n}{2} \rfloor} \subset \ldots \subset I_2 \subset I_1 \subset I_0$, where I_k is generated by crossingless planar tangles with $\geq k$ turnbacks.

2. In this question, we consider the relationship between the category $\mathrm{Cob}(n)$ described in Cooper and Krushkal's paper and the TQFT $\mathcal{A}$ appearing in Bar-Natan's paper. For simplicity we let α (the parameter appearing on page 7 of C-K) be 0 and tensor everything in sight with $\mathbb{Q}$. (So 2 is invertible.)

   (a) Let B be a crossingless closed curve in the plane, which we view as an element of $\mathrm{Cob}(0)$. Show that as a graded vector space, $\mathrm{Mor}(B, \emptyset)$ can be naturally identified with $\mathcal{A}(B)$.

   (b) Now let B_1 and B_2 be two crossingless planar n-tangles, which we view as elements of $\mathrm{Cob}(n)$. Show that $\mathrm{Mor}(B_1, B_2)$ can be identified with $\mathcal{A}(\overline{B_1 B_2 r})$, where $\overline{B_1 B_2^r}$ is the crossingless planar diagram shown below.

3. Consider the Khovanov chain complex associated to a 2-strand braid. Let $\mathbf{1}$ be the identity element of TL_2, and let and $\mathbf{B}_1$ be the basis element corresponding to the other crossingless planar tangle. Let $C(\sigma_1)$ and $C(\sigma_1^{-1})$ be the chain complexes associated to positive and negative crossings, as on page 8 of Cooper-Krushkal.

   (a) Show that $C(\sigma) \otimes C(\sigma^{-1})$ is isomorphic to a chain complex of the form

   $$q^{-1}\mathbf{B_1} \xrightarrow{(*\,1\,*)^T} \mathbf{1} \oplus q^{-1}\mathbf{B_1} \oplus q\mathbf{B_1} \xrightarrow{(*\,*\,1)} q\mathbf{B_1}$$

   What are non-identity morphisms in the complex (those labeled by $*$'s)? Check that $d^2 = 0$.

   (b) Using Gaussian elimination, show that any complex of the form above (regardless of what the $*$'s are) is isomorphic to a complex consisting of the single object $\mathbf{1}$ in homological grading 0. Explain why this proves that the Khovanov complex is invariant under the Reidemeister II move (as in Bar-Natan's paper.)

4. As in the previous problem, we work with 2-strand braids. Now we consider $C(\sigma_1^n)$.

(a) Show that $C(\sigma_1^2)$ is homotopy equivalent to a complex of the form

$$q^2\mathbf{1} \to q^3\mathbf{B}_1 \to q^5\mathbf{B}_1.$$

What are the boundary maps?

(b) By induction, show that $C(\sigma_1^n)$ is homotopy equivalent to a complex of the form

$$q^{2n}\mathbf{1} \to q^{2n+1}\mathbf{B}_1 \to q^{2n+3}\mathbf{B}_1 \to \ldots \to q^{3n-1}\mathbf{B}_1.$$

What are the boundary maps?

(c) Compute $Kh(\overline{\sigma_1^n})$.

5. Now we work with 3-strand braids. Compute $C(\sigma_1\sigma_2\sigma_1)$. (You may find it helpful to use the mutiplication table from exercise 1.) Apply Gaussian elimination to construct a complex in which no component of the differential is the identity. Do the same thing for $C(\sigma_2\sigma_1\sigma_2)$ and check that the two resulting complexes are isomorphic. Explain why this proves invariance of Khovanov homology under the Reidemeister III move.

6. We will view the Hecke algebra H_n as an algebra over $\mathbb{C}$ generated (as an algebra) by $T_1, \ldots T_{n-1}$, with relations

$$\begin{aligned} T_i^2 &= (q^2-1)T_i + q^2 \\ T_iT_jT_i &= T_jT_iT_j \quad \text{for } (|i-j|=1) \\ T_iT_j &= T_jT_i \qquad \text{for } (|i-j|>1). \end{aligned}$$

In comparison with the presentation in Jones' paper, his $q^{1/2}$ is our q, and his g_i is our T_i. We view the Jones-Ocneanu trace as a map $\text{tr} : H_n \to \mathbb{C}[a^{\pm 1}, q^{\pm 1}]$, where in the notation of Prop. 6.2 of Jones' paper, $a = (q\lambda)^{1/2}$ and $q = q^{1/2}$. With this normalization, the skein relation for the HOMFLY polynomial reads

$$a^{-1}P(L_+) - aP(L_-) = (q-q^{-1})P(L_0).$$

Finally, we use a renormalized form of the trace:

$$\text{Tr}'(\mathbf{B}) = \left(\frac{a-a^{-1}}{q-q^{-1}}\right)^n \text{tr}(\mathbf{B})$$

for $\mathbf{B} \in H_n$.

(a) Let $\mathbf{B}_i = q - q^{-1}T_i$. Express the presentation for H_n given above in terms of the $\mathbf{B}_i$. Deduce that there is a surjective homomorphism $\pi : H_n \to TL_n$.

(b) For $n = 3$, show that the kernel of π is a one-dimensional vector space generated by $\mathbf{B}_w = \mathbf{B}_1\mathbf{B}_2\mathbf{B}_1 - \mathbf{B}_1$. Write out the multiplication table on H_3 in terms of the basis $\langle \mathbf{1}, \mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_1\mathbf{B}_2, \mathbf{B}_2\mathbf{B}_1, \mathbf{B}_w \rangle$.

(c) If $i : H_n \to H_{n+1}$ is the inclusion, show that Tr' satisfies the following relations:

$$\mathrm{Tr}'(i(\mathbf{B})) = \{0\}\,\mathrm{Tr}'(\mathbf{B})$$
$$\mathrm{Tr}'(i(\mathbf{B})\mathbf{B}_n) = \{1\}\,\mathrm{Tr}'(\mathbf{B})$$
$$\mathrm{Tr}'(\mathbf{B}\mathbf{B}') = \mathrm{Tr}'(\mathbf{B}'\mathbf{B})$$

where $\{i\} = \dfrac{aq^{-i} - a^{-1}q^i}{q - q^{-1}}$.

Use these relations, together with the relations in H_3 to evaluate Tr' on each of the basis elements above. Express your answers in terms of $\{i\}$ for various values of i.

(d) Let $p_i : \mathbb{C}[a^{\pm 1}, q^{\pm 1}] \to \mathbb{C}[q^{\pm 1}]$ be the homomorphism given by $p_i(a) = q^i$. Show that $p_1(\mathrm{Tr}'(\mathbf{B})) = 0$, where $\mathbf{B}$ is any nontrivial product of the $\mathbf{B}_j$'s, and that $p_2(\mathrm{Tr}'(\mathbf{B})) = (q + q^{-1})^{n(\pi(\mathbf{B}))}$, where $n(\pi(\mathbf{B}))$ is the number of circles in the braid closure of $\pi(\mathbf{B})$.

(e) (Optional, for those who know about Schubert varieties.) How are your answers to part c) related to the topology of Schubert varieties in the variety of complete flags in $\mathbb{C}^3$?