MONDAY EXERCISE 2

Consider the ideal $I = \langle x + y + z \rangle \subseteq \mathbb{C}[x, y, z]$.

(1) Describe the matroid $\mathrm{Mat}_1(I)$ on the ground set $E_1 = \{x, y, z\}$. A set S is independent in $\mathrm{Mat}_1(I)$ if there is no polynomial in I of degree one with support contained in S.
   (a) List the independent sets of $\mathrm{Mat}_1(I)$.
   (b) List the bases.
   (c) What is the rank?
   (d) List the circuits.
   (e) List the flats, and draw the lattice of flats.

(2) Repeat the previous question for the matroid $\mathrm{Mat}_2(I)$ on the ground set $E_2 = \{x^2, xy, xz, y^2, yz, z^2\}$.

(3) Do your answers change if we replace I by $\langle ax + by + cz \rangle$ where a, b, c are arbitrary complex numbers?

1. Supplementary Exercises

(1) Describe and prove at least one cryptomorphism between different descriptions of a matroid. This means, for example, how to go between the bases and the circuits of a matroid, and showing that if the first set satisfies the basis axioms, then the second set obeys the circuit axioms.

(2) The *Fano matroid* is the matroid on $E = \{1, 2, 3, 4, 5, 6, 7\}$ with circuits $\{123, 147, 156, 246, 257, 345, 367\}$.
   (a) Check that this obeys the circuit axioms.
   (b) What are the bases of this matroid? What is the rank?
   (c) Show that this is not realizable over a field of characteristic not equal to two.

   The Fano matroid is actually realizable over $\mathbb{F}_2$: it is the matroid given by all nonzero vectors in $\mathbb{F}_2^3$ (or equivalently in $\mathbb{P}^2_{\mathbb{F}_2}$). Google for pictures!

(3) A *cocircuit* of a matroid is a circuit of the dual matroid. List all the cocircuits for the matroids you have met so far (in the main exercise, or above). For a realizable matroid, a cocircuit is the set of vectors not on a hyperplane spanned by some of the vectors. Verify this in your examples.

(4) Show that if A is a $d \times n$ matrix of rank d, and B is an $(n-d) \times n$ matrix with rows a basis for $\ker(A)$, then the minor of A indexed by a subset $B \subset \{1, \dots, n\}$ of size d equals the minor of B indexed by $\{1, \dots, n\} \setminus B$. This shows that the matroid given by the columns of B is the dual of the matroid given by the columns of A.

(5) Show that the bases of matroids obey the *strong basis exchange axiom*: For all $B_1, B_2 \in \mathcal{B}(M)$, and all $i \in B_1 \setminus B_2$, there is $j \in B_2 \setminus B_1$ with $B_1 \setminus \{i\} \cup \{j\}$ *and* $B_2 \setminus \{j\} \cup \{i\}$ in $\mathcal{B}(M)$.