MONDAY EXERCISE 3
ALGEBRA OF SEMIRINGS AND THEIR MODULES

Let $M$ be the free $T$-module generated by elements $e_1, e_2, e_3$. Let $\kappa$ be the module congruence on $M$ generated by the relations

$$e_1 + e_2 \sim e_1 + e_3 \sim e_2 + e_3.$$

(1) Find a minimal generating set for the submodule $(M/\kappa)^\vee \subset T^3$. (Do you recognize these as the circuits of a matroid?)

(2) Using the generating set, find a presentation for the double dual $(M/\kappa)^{\vee\vee}$. Is it isomorphic to $M/\kappa$?

(3) Now consider the semiring $S = T[e_1, e_2, e_3]$ and the congruence $\langle \kappa \rangle$ on $S$ generated by $\kappa$. Describe the sets

$\text{Hom}(S, T)$, and $\text{Hom}(S/\langle \kappa \rangle, T)$.

(These Homs denote homomorphisms of semirings.)

(4) Does there exist a larger semiring congruence $\gamma \supset \langle \kappa \rangle$ such that

$\text{Hom}(S/\langle \kappa \rangle, T) = \text{Hom}(S/\gamma, T)$?
Supplementary exercises

(1) Check that the intersection of two congruences on a module is a congruence. Check that the intersection of two congruences on a semiring is a congruence.

(2) Let $M$ be a $T$-module and $Z \subset M \times M$ a submodule. Show that the transitive closure of $Z$ is also a submodule, and hence it is a module congruence. Now suppose $S$ is a semiring and $W \subset S \times S$ is a subsemiring. Show that the transitive closure of $W$ is subsemiring and hence a semiring congruence.

(3) Let $S$ be a semiring. Show that the semiring congruence on $S$ generated by a relation $a \sim b$ is given by taking the sub-semiring of $S \times S$ generated by this pair and then taking the transitive closure of this set.

(4) Consider the $T$-module $M = \{(a, b) | a < b\} \subset T^2$. Find a presentation for the dual $M^\vee$ as a quotient of $T^2$.

(5) Recall that a join-semilattice (in the sense from order theory) is a partially ordered set such that every pair of elements $a, b$ has a unique least upper bound (called the ‘join’ of $a$ and $b$), and a lattice is a partially ordered set such that each pair of elements has a unique least upper bound and a unique greatest lower bound (called the ‘meet’). A morphism of join-semilattices is an order-preserving map that sends joins to joins.

(a) Show that there is an equivalence of categories between join-semilattices and $\mathbb{B}$-modules.

(b) Given a matroid $M$ on ground set $E$, the rank of a subset $X \subset E$ is smallest cardinality of an independent set containing $X$, and a flat is a subset of $E$ such that adding any additional element increases the rank. Show that the flats of $M$ form a lattice.

(c) Given an element $v = \sum_i v_i e_i \in \mathbb{B}^n$, recall that $\mathcal{B}(v)$ is the module congruence on $\mathbb{B}^n$ generated by the bend relations $v \sim \sum_{i \neq j} v_i e_i$ for $j = 1 \ldots n$. Show that the lattice of flats of $M$, as a join-semilattice, corresponds to the $\mathbb{B}$-module $\mathbb{B}^E/\mathcal{B}(c_1, c_2, \ldots)$, where $c_i$ are the circuits of $M$.

(6) Given $f \in T[x_1, x_2, x_3]$ and an injective homomorphism of semirings

$$\varphi : T[x_1, x_2, x_3] \to T[y_1, y_2, y_3, y_4]$$

sending monomials to monomials, show that $\varphi_* \mathcal{B}(f) = \mathcal{B}(\varphi(f))$. 