MONDAY EXERCISE 3
ALGEBRA OF SEMIRINGS AND THEIR MODULES

Let M be the free $\mathbb{T}$-module generated by elements e_1, e_2, e_3. Let κ be the module congruence on M generated by the relations

$$e_1 + e_2 \sim e_1 + e_3 \sim e_2 + e_3.$$

(1) Find a minimal generating set for the submodule $(M/\kappa)^\vee \subset \mathbb{T}^3$. (Do you recognize these as the circuits of a matroid?)

(2) Using the generating set, find a presentation for the double dual $(M/\kappa)^{\vee\vee}$. Is it isomorphic to M/κ?

(3) Now consider the semiring $S = \mathbb{T}[e_1, e_2, e_3]$ and the congruence $\langle\kappa\rangle$ on S generated by κ. Describe the sets

$$\mathrm{Hom}(S, \mathbb{T}), \text{ and } \mathrm{Hom}(S/\langle\kappa\rangle, \mathbb{T}).$$

(These Homs denote homomorphisms of semirings.)

(4) Does there exist a larger semiring congruence $\gamma \supset \langle\kappa\rangle$ such that

$$\mathrm{Hom}(S/\langle\kappa\rangle, \mathbb{T}) = \mathrm{Hom}(S/\gamma, \mathbb{T})?$$

Supplementary exercises

(1) Check that the intersection of two congruences on a module is a congruence. Check that the intersection of two congruences on a semiring is a congruence.

(2) Let M be a $\mathbb{T}$-module and $Z \subset M \times M$ a submodule. Show that the transitive closure of Z is also a submodule, and hence it is a module congruence. Now suppose S is a semiring and $W \subset S \times S$ is a subsemiring. Show that the transitive closure of W is subsemiring and hence a semiring congruence.

(3) Let S be a semiring. Show that the semiring congruence on S generated by a relation $a \sim b$ is given by taking the sub-semiring of $S \times S$ generated by this pair and then taking the transitive closure of this set.

(4) Consider the $\mathbb{T}$-module $M = \{(a, b)|a < b\} \subset \mathbb{T}^2$. Find a presentation for the dual $M^\vee$ as a quotient of $\mathbb{T}^2$.

(5) Recall that a *join-semilattice* (in the sense from order theory) is a partially ordered set such that every pair of elements a, b has a unique least upper bound (called the 'join' of a and b), and a *lattice* is a partially ordered set such that each pair of elements has a unique least upper bound and a unique greatest lower bound (called the 'meet'). A morphism of join-semilattices is an order-preserving map that sends joins to joins.

   (a) Show that there is an equivalence of categories between join-semilattices and $\mathbb{B}$-modules.

   (b) Given a matroid M on ground set E, the rank of a subset $X \subset E$ is smallest cardinality of an independent set containing X, and a *flat* is a subset of E such that adding any additional element increases the rank. Show that the flats of M form a lattice.

   (c) Given an element $v = \sum_i v_i e_i \in \mathbb{B}^n$, recall that $\mathcal{B}(v)$ is the module congruence on $\mathbb{B}^n$ generated by the bend relations $v \sim \sum_{i \neq j} v_i e_i$ for $j = 1 \ldots n$. Show that the lattice of flats of M, as a join-semilattice, corresponds to the $\mathbb{B}$-module $\mathbb{B}^E/\mathcal{B}(c_1, c_2, \ldots)$, where c_i are the circuits of M.

(6) Given $f \in \mathbb{T}[x_1, x_2, x_3]$ and an injective homomorphism of semirings

$$\varphi : \mathbb{T}[x_1, x_2, x_3] \to \mathbb{T}[y_1, y_2, y_3, y_4]$$

sending monomials to monomials, show that $\varphi_* \mathcal{B}(f) = \mathcal{B}(\varphi(f))$.