MONDAY EXERCISE 4

Consider the ideal $I = \langle x_1 + x_2 + x_3 + x_4, x_1 + 2x_2 - x_3 + x_4 \rangle \subseteq S := \mathbb{C}[x_1, \dots, x_4]$.

(1) Compute the degree-one matroid M_1 of I_1 (i.e., describe all bases or circuits, or another equivalent description).

(2) Draw the Bergman fan of M_1 in $\mathbb{R}^4/\mathbb{R}\mathbf{1}$.

(3) Draw the variety $V(I) \subseteq \text{trop}(\mathbb{P}^n)$.

(4) Do your answer depend on the coefficients of the two generators of I?

1. Supplementary Exercises

(1) Let $M = U_{n,n}$ be the uniform matroid of rank n on n elements. This has one basis, which is the entire ground set $E = \{1, \dots, n\}$. Show that the fine fan structure on $\mathrm{trop}(M)$ is the subdivision given by the hyperplane arrangement with hyperplanes $\mathbf{e}_i - \mathbf{e}_j$ and $\mathbf{e}_i$. Show that the Bergman fan of any matroid on $\{1, \dots, n\}$ is a subfan of $\mathrm{trop}(M)$.

(2) Let $I = \langle x - y \rangle \subseteq \mathbb{C}[x, y]$. Show that $\mathrm{trop}(I) \subseteq \mathbb{B}[x, y]$ is not finitely generated.

(3) Let $I = \langle x_1 + x_3 + x_4, x_2 + x_3 + ax_4 \rangle \subseteq \mathbb{C}[x_1, x_2, x_3, x_4]$, where $a \in \mathbb{C}$. Show that $\mathrm{trop}(I) \subseteq \mathbb{B}[x_1, x_2, x_3, x_4]$ is constant for a outside a countable subset of $\mathbb{C}$, but there are an infinite number of different tropicalizations $\mathrm{trop}(I)$ as a varies over all of $\mathbb{C}$.