THURSDAY EXERCISE 1

Let $V = \mathbb{T}^4$, and let $U = \mathrm{Hom}(V, \mathbb{T})$ be its linear dual. Let

$$p = \sum_{i<j} p_{ij} e_i \wedge e_j \in \wedge^2 V$$

be a valuated matroid whose underlying matroid is the uniform matroid $U_{2,4}$ (i.e., all components of p are nonvanishing).

(1) Show that the image of $(- \wedge p)^\vee : \wedge^{d+1} U \to U$ is spanned by the circuit vectors of p.

(2) Use the circuits to give a presentation of the associated quotient module $\mathbb{T}^4 \twoheadrightarrow Q_p$.

(3) Now, by wedging generating relations with basis vectors, find a presentation for $\wedge^2 Q_p$. Using this presentation, show that $\wedge^2 Q_p \cong \mathbb{B}$.

(4) Next, try the same for the valuated matroid

$$(7)e_1 \wedge e_2 + (2)e_1 \wedge e_3 + (4)e_1 \wedge e_4 + (0)e_2 \wedge e_3 + (2)e_2 \wedge e_4 + (\infty)e_3 \wedge e_4.$$

Supplementary exercises

(1) If $p \in \wedge^c V$ and $q \in \wedge^d V$ are valuated matroids, show that their stable sum is represented by $p \wedge q$.

(2) Let $V = \mathbb{T}^n$, with basis $\{e_i\}$. Given $p = \sum_{|A|=d} p_A e_A \in \wedge^d V$, show that the circuit vectors of p^* can be written in the form

$$\sum_{i \notin B} p_{B \cup i} e_i \in V$$

for $|B| = d - 1$.

(3) Find an example of a valuated matroid of rank d on $\{1, \ldots, n\}$ and two sets $A, A' \subset \{1, \ldots, n\}$ of size $d + 1$ that both produce the same circuit vector (up to scaling).