THURSDAY EXERCISE 3
UNIVERSAL TROPICALIZATION

Let k be a field with valuation $v : k \to \mathbb{T}$, and let A be a k-algebra. We write $A^{\times}$ for the multiplicative monoid of A.

(1) Show that the kernel of the evaluation map $\mathrm{ev} : k[A^{\times}] \to A$ is spanned by the elements of the form
- $\lambda x_a - x_{\lambda a}$ for $a \in A$ and $\lambda \in k$;
- $x_a + x_b + x_c$ for $a, b, c \in A$ with $a + b + c = 0$.

(2) Let $J = \mathrm{trop}(\ker \mathrm{ev})$. Now show that the congruence $\mathcal{B}(J)$ is generated by the congruences $\mathcal{B}(v(\lambda)x_a - x_{\lambda a})$ and $\mathcal{B}(x_a + x_b + x_c)$ for $a + b + c = 0$.

(3) Show that there is a multiplicative map

$$\mathrm{Val}_{univ} : A \to \mathbb{T}[A^{\times}]/\mathcal{B}(J)$$

such that any valuation $w : A \to \mathbb{T}$ extending the valuation on k has a unique factorization

$$A \to \mathbb{T}[A^{\times}]/\mathcal{B}(J) \to \mathbb{T}.$$

(The map Val_{univ} is the *universal valuation* on A.)

Supplementary exercises

(1) Let Z_1 and Z_2 be monoids and suppose we are given embeddings
$$\alpha_i : \operatorname{spec} A \hookrightarrow \operatorname{spec} k[Z_i].$$
Given a homomorphism of monoids $\phi : Z_1 \to Z_2$ that induces a morphism $\operatorname{spec} k[Z_2] \to \operatorname{spec} k[Z_1]$ commuting with the embeddings, show that there is an induced morphism of troicalizations.
$$\operatorname{trop}_{\alpha_2}(\operatorname{spec} A) \to \operatorname{trop}_{\alpha_1}(\operatorname{spec} A).$$