THURSDAY EXERCISE 4

The Cox ring of $\mathbb{P}^n$ is the polynomial ring $K[x_0, \dots, x_n]$ with the standard grading $\deg(x_i) = 1$.

(1) A homogeneous ideal $I \subset S := \overline{\mathbb{R}}[x_0, \dots, x_n]$ is weakly tropical if $(IS_{x_i})_0 \subseteq (\overline{\mathbb{R}}[x_0, \dots, x_n]_{x_i})_0 \cong \overline{\mathbb{R}}[y_0, \dots, \hat{y}_i, \dots, y_n]$ is a tropical ideal. Show that $\langle x_0 \oplus x_1, x_0^3, x_1^3 \rangle$ is weakly tropical for $\mathbb{P}^1$, but not tropical.

(2) The tropicalization $\operatorname{trop}(\mathbb{P}^2)$ can be obtained by gluing together three affine charts $\operatorname{trop}(\mathbb{A}^2) = \overline{\mathbb{R}}^2$. Consider the ideal

$$\operatorname{trop}(\langle x_0^2 + x_0x_1 + x_0x_2 + 5x_1x_2 \rangle) \subset \overline{\mathbb{R}}[x_0, x_1, x_2],$$

where $\mathbb{Q}$ has the 5-adic valuation. Compute the localizations $I_i = (I\overline{\mathbb{R}}[x_0, x_1, x_2]_{x_i})_0$ for each $0 \leq i \leq 2$, and draw the corresponding tropical varieties $V(I_i) \subset \overline{\mathbb{R}}^2$. Glue these to get a subvariety of $\operatorname{trop}(\mathbb{P}^2)$. Compare this with the result of computing $V(I)$ in $\overline{\mathbb{R}}^3$, and taking the image in $\operatorname{trop}(\mathbb{P}^2) = (\overline{\mathbb{R}}^3 \setminus (\infty, \infty, \infty))/\mathbb{R}(1, 1, 1)$ (it should be the same!).

The Cox ring of $\mathbb{P}^1 \times \mathbb{P}^1$ is $K[x_0, x_1, y_0, y_1]$ with the grading $\deg(x_i) = (1, 0)$, $\deg(y_i) = (0, 1)$.

(3) Repeat the previous exercise for the ideal

$$\operatorname{trop}(\langle x_0y_0 + x_1y_0 + x_0y_1 + x_1y_1 \rangle) \subseteq \overline{\mathbb{R}}[x_0, x_1, y_0, y_1].$$

Supplementary exercises

(1) (For people who have seen toric varieties). The tropical toric variety associated to a rational polyhedral fan $\Sigma \subset N_{\mathbb{R}}$ is defined as follows: For $\sigma \in \Sigma$, the tropicalization U_σ^{trop} is $\text{Hom}_{sg}(\sigma^\vee \cap M, \overline{\mathbb{R}})$. If τ is a face of σ, then $\sigma^\vee$ is a submonoid of $\tau^\vee$, so there is a map from U_τ^{trop} to $U\sigma^{\text{trop}}$ given by $f \mapsto f|_{\sigma^\vee}$. The tropical toric variety $\text{trop}(X_\Sigma)$ is obtained by gluing together the U_σ^{trop} along these maps.

   (a) Draw the tropical toric variety $\text{trop}(\mathbb{P}(1,2,3))$. This has the fan with three rays $\{(1,0),(0,1),(-2,-3)\}$, each pair of which spans a two-dimensional cone.
   (b) Compute the tropicalization of the variety $V(x_1^6 + 3x_1^4x_2 + 2x_2^3 - x_1^3x_3 + 7x_1x_2x_3) \subseteq \mathbb{P}(1,2,3)$ under the trivial valuation.
   (c) Compare this with tropicalization of this as an affine variety in $\mathbb{A}^3$.
   (d) Generalise what you observe (this can be viewed as tropicalizing the Cox construction of a toric variety as a GIT quotient of affine space).

(2) (For people who like more abstraction). Look at Lorscheid's paper `https://arxiv.org/abs/1907.01037`. Understand the tropical hyperfield from this perspective, and how this relates to scheme-theoretic tropicalization.