TUESDAY EXERCISE 1

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 7 & -1 \\ 0 & 1 & 3 & 5 \end{pmatrix}.$$

Let M be the valuated matroid given by setting $\rho\colon \binom{\{1,2,3,4\}}{2} \to \overline{\mathbb{R}}$ to be $\rho(I) = \mathrm{val}(\det(A_I))$, where A_I is the submatrix of A with columns indexed by I, and val is the 2-adic valuation on $\mathbb{Q}$.

(1) Write down these minors explicitly.
(2) What is the underlying (non-valuated) matroid of M?
(3) List the valuated circuits of M. Verify one example of valuated circuit elimination for your list.
(4) Give an example of a vector of M that is not a circuit.
(5) Draw the collection of all vectors of M in $\overline{\mathbb{R}}^4/\mathbb{R}\mathbf{1}$.

1. Supplementary Exercises

(1) Repeat the main exercise with the p-adic valuation on $\mathbb{Q}$, where p is an odd prime of your choice.

(2) Let M be a valuated matroid with basis function $\rho\colon \binom{E}{d}$. Show that every circuit of M has the form

$$\sum_{i \in C} \rho(C \setminus i)\mathbf{e}_i$$

for some set $C = B \cup \{j\}$, where B is a basis of M.

(3) Describe the cryptomorphism from circuits to basis functions. In other words, explain how to write down $\rho\colon \binom{E}{d} \to \overline{\mathbb{R}}$ given a collection $\mathcal{C} \subset \overline{\mathbb{R}}^E$.

(4) Valuated matroids also have duality. Show that if $\rho\colon \binom{E}{d} \to \overline{\mathbb{R}}$ is a basis function, then so is $\rho'\colon \binom{E}{|E|-d} \to \overline{\mathbb{R}}$ given by

$$\rho'(B) = \rho(E \setminus B).$$

(5) The cocircuits of a valuated matroid M are the circuits of the dual matroid M^*. Show that if $\mathbf{v} \in \overline{\mathbb{R}}^E$ is *tropically orthogonal* to all vectors of M (the minimum in $\min_i(v_i + u_i)$ is achieved at least twice for all vectors $\mathbf{u}$) then $\mathbf{v}$ is a tropical linear combination of cocircuits of M.

(6) Describe all valuated matroids of rank 2 on $E = \{1, 2, 3, 4\}$ with underlying matroid the uniform matroid (so $\rho(B) \neq \infty$ for all $B \in \binom{E}{2}$). You can regard $\rho\colon \binom{E}{2} \to \overline{\mathbb{R}}$ as a vector in $\overline{\mathbb{R}}^6$. Describe this subset. What does this have to do with the Grassmannian?