TUESDAY EXERCISE 2
TROPICAL IDEALS, PART 1

Consider the ideal \( I = \langle x + 8y + 2z \rangle \subset \mathbb{Q}[x, y, z] \) and equip \( \mathbb{Q} \) with the 2-adic valuation, and let \( J = \text{trop}(I) \).

(1) Describe the valued matroid \( M_1(J) \) on \( E_1 = \{x, y, z\} \).

(a) List the valued bases.

(b) List the valued circuit vectors.

(2) Now do the same for the valued matroid \( M_2(J) \) on \( E_2 = \{x^2, y^2, z^2, xy, xz, yz\} \).

(3) Check that, under multiplication by \( x, y \) or \( z \), the compatibility condition for being an ideal holds between \( M_1(J) \) and \( M_2(J) \).

(4) Is the tropical ideal \( J \) finitely generated?
Supplementary exercises

(1) Let $k$ be a valued field and $I \subset k[x_0, \ldots, x_n]$ a homogeneous prime binomial ideal.

(a) Show that the tropical ideal $\text{trop}(I)$ is not finitely generated.

(b) In contrast, show that the congruence $\mathcal{B}(\text{trop}(I))$ is finitely generated.

(2) Consider the ideals $I = \langle (x + y)(x + z)(y + z) \rangle$ and $J = \langle (x + y + z)(xy + yz + xz) \rangle$ in $\mathbb{C}[x, y, z]$, and their tropicalizations using the trivial valuation on $\mathbb{C}$.

(a) Show that these two tropical ideals are not equal.

(b) Show that they have the same variety.

(3) Consider the semiring $T[x_0, \ldots, x_n]$ and let $y_0, \ldots, y_N$ be the set of degree $d$ monomials ($N = \binom{n+d}{n} - 1$). The degree $d$ tropical Veronese map is the surjective homomorphism

$$
\varphi_d : T[y_0, \ldots, y_N] \to T[x_0, \ldots, x_n]
$$

that sends each $y_j$ to the corresponding monomial in the $x_i$ variables. Find a tropical ideal $I \subset T[x_0, \ldots, x_n]$ such that the Veronese map descends to an isomorphism $T[y_0, \ldots, y_N]/\mathcal{B}(I) \cong T[x_0, \ldots, x_n]$.

(4) (a) Let $m$ be a monomial in the $x_i$ and consider the localization map $\varphi : T[x_0, \ldots, x_n] \to T[x_0, \ldots, x_n, m^{-1}]$. Given a tropical ideal $I \subset T[x_0, \ldots, x_n]$, show that $\varphi_* I$ (the ideal generated by the image of $I$) is a tropical ideal.

(b) Given a homogeneous tropical ideal $J$ in the graded semiring $T[x_0^{\pm 1}, x_1, \ldots, x_n]$, show the restriction to the degree 0 component $T[x_0^{\pm 1}, x_1, \ldots, x_n]_0 \cong T[x_1/x_0, \ldots, x_n/x_0]$ sends $J$ to a tropical ideal.

(c) Given an inhomogeneous tropical ideal $I \subset T[x_1, \ldots, x_n]$, let $\tilde{I} \subset T[x_0, x_1, \ldots, x_n]$ denote its homogenization with respect to $x_0$. Show that $\tilde{I}$ is a tropical ideal.

(5) Let $I_p$ be the point ideal associated with a point $p \in \mathbb{TP}^n$; i.e., the set of all homog Show that $I_p$ is the tropicalization of the ideal $J_{\tilde{p}}$ of functions vanishing a point $\tilde{p}$, where $\tilde{p} \in \mathbb{P}_k^n$ is any point that tropicalizes to $p$. 