TUESDAY EXERCISE 2
TROPICAL IDEALS, PART 1

Consider the ideal $I = \langle x + 8y + 2z \rangle \subset \mathbb{Q}[x, y, z]$ and equip $\mathbb{Q}$ with the 2-adic valuation, and let $J = \mathrm{trop}(I)$.

(1) Describe the *valuated matroid* $M_1(J)$ on $E_1 = \{x, y, z\}$.

   (a) List the valuated bases.

   (b) List the valuated circuit vectors.

(2) Now do the same for the valuated matroid $M_2(J)$ on $E_2 = \{x^2, y^2, z^2, xy, xz, yz\}$.

(3) Check that, under multiplication by x, y or z, the compatibility condition for being an ideal holds between $M_1(J)$ and $M_2(J)$.

(4) Is the tropical ideal J finitely generated?

Supplementary exercises

(1) Let k be a valued field and $I \subset k[x_0, \dots, x_n]$ a homogeneous prime binomial ideal.

   (a) Show that the tropical ideal $\operatorname{trop}(I)$ is not finitely generated.

   (b) In contrast, show that the congruence $\mathcal{B}(\operatorname{trop}(I))$ is finitely generated.

(2) Consider the ideals $I = \langle (x+y)(x+z)(y+z) \rangle$ and $J = \langle (x+y+z)(xy+yz+xz) \rangle$ in $\mathbb{C}[x,y,z]$, and their tropicalizations using the trivial valuation on $\mathbb{C}$.

   (a) Show that these two tropical ideals are not equal.

   (b) Show that they have the same variety.

(3) Consider the semiring $\mathbb{T}[x_0, \dots, x_n]$ and let $y_0, \dots, y_N$ be the set of degree d monomials ($N = \binom{n+d}{n} - 1$). The degree d *tropical Veronese map* is the surjective homomorphism

$$\varphi_d : \mathbb{T}[y_0, \dots, y_N] \to \mathbb{T}[x_0, \dots, x_n]$$

that sends each y_j to the corresponding monomial in the x_i variables. Find a tropical ideal $I \subset \mathbb{T}[y_0, \dots, y_N]$ such that the Veronese map descends to an isomorphism $\mathbb{T}[y_0, \dots, y_N]/\mathcal{B}(I) \cong \mathbb{T}[x_0, \dots, x_n]$.

(4) (a) Let m be a monomial in the x_i and consider the localization map $\varphi : \mathbb{T}[x_0, \dots, x_n] \to \mathbb{T}[x_0, \dots, x_n, m^{-1}]$. Given a tropical ideal $I \subset \mathbb{T}[x_0, \dots, x_n]$, show that $\varphi_* I$ (the ideal generated by the image of I) is a tropical ideal.

   (b) Given a homogeneous tropical ideal J in the graded semiring $\mathbb{T}[x_0^{\pm 1}, x_1, \dots, x_n]$, show the restriction to the degree 0 component $\mathbb{T}[x_0^{\pm 1}, x_1, \dots, x_n]_0 \cong \mathbb{T}[x_1/x_0, \dots, x_n/x_0]$ sends J to a tropical ideal.

   (c) Given an inhomogeneous tropical ideal $I \subset \mathbb{T}[x_1, \dots, x_n]$, let $\widetilde{I} \subset \mathbb{T}[x_0, x_1, \dots, x_n]$ denote its homogenization with respect to x_0. Show that $\widetilde{I}$ is a tropical ideal.

(5) Let I_p be the point ideal associated with a point $p \in \mathbb{TP}^n$; i.e., the set of all homog Show that I_p is the tropicalization of the ideal $J_{\widetilde{p}}$ of functions vanishing a point $\widetilde{p}$, where $\widetilde{p} \in \mathbb{P}^n_k$ is any point that tropicalizes to p.