TUESDAY LECTURE 3

Consider the ideal $I = \langle 24x^2 - xy + 8y^2 + 6xz - 10yz + 11z^2 \rangle \subseteq \mathbb{Q}[x, y, z]$, where $\mathbb{Q}$ has the 2-adic valuation.

(1) Draw the Gröbner complex of $\operatorname{trop}(I) \subseteq \overline{\mathbb{R}}[x, y, z]$.
(2) What is the Hilbert function of $\operatorname{trop}(I)$? What is the Hilbert polynomial? What is the dimension of $\operatorname{trop}(I)$?
(3) Give a chain of tropical ideals of length 5 containing $\operatorname{trop}(I)$. How many different varieties are there of ideals in your chain?

1. Supplementary Exercises

(1) If $I \subseteq \overline{\mathbb{R}}[x_0, \dots, x_n]$ is a homogeneous tropical ideal, then $\text{in}_{\mathbf{w}}(I) \subseteq \mathbb{B}[x_0, \dots, x_n]$ is a homogeneous tropical ideal, so each degree d part defines a (non-valuated) matroid $\text{Mat}(\text{in}_{\mathbf{w}}(I))$.
   (a) We define the $\mathbf{w}$ weight of a basis B of the degree d matroid to be $\sum_{i \in B} w_i - \rho(B)$, where ρ is the basis valuation function for the valuated matroid of $\text{Mat}(I_d)$. Show that the bases of $\text{Mat}(\text{in}_{\mathbf{w}}(I)_d)$ are the bases of $\text{Mat}(I_d)$ of maximal weight.
   (b) Show that the circuits of $\text{Mat}(\text{in}_{\mathbf{w}}(I)_d)$ are those initial terms of circuits of $\text{Mat}(I_d)$ that have minimal support.

(2) (If you know what a matroid polytope is). Compare the matroid polytope of the underlying matroid of $\text{Mat}(I_d)$ and the matroid polytope of $\text{Mat}(\text{in}_{\mathbf{w}}(I)_d)$. If you know what a regular subdivision is, how is that relevant here?

(3) Show that a tropical ideal is not necessarily determined by a finite set of degrees. Hint: Monday Exercise 4, supplementary 3.

(4) Tropical ideals obey the weak Nullstellensatz: $V(I) = \emptyset$ if and only if $I = \langle 0 \rangle$. Give an example to show that this is false for an arbitrary ideal in $\overline{\mathbb{R}}[x_1, \dots, x_n]$.

(5) Let I be a non-homogeneous tropical ideal in $\overline{\mathbb{R}}[x]$. This means that for any finite collection E of monomials in $\overline{\mathbb{R}}[x]$, the set of polynomials in I supported in E is the set of vectors of a valuated matroid. Let f be a polynomial in I of lowest degree. Show that $V(I) = V(f)$.