Consider the ideal 

\[ I = \langle 24x^2 - xy + 8y^2 + 6xz - 10yz + 11z^2 \rangle \subseteq \mathbb{Q}[x, y, z], \]

where \( \mathbb{Q} \) has the 2-adic valuation.

(1) Draw the Gröbner complex of \( \text{trop}(I) \subseteq \mathbb{R}[x, y, z] \).

(2) What is the Hilbert function of \( \text{trop}(I) \)? What is the Hilbert polynomial? What is the dimension of \( \text{trop}(I) \)?

(3) Give a chain of tropical ideals of length 5 containing \( \text{trop}(I) \). How many different varieties are there of ideals in your chain?
1. Supplementary Exercises

(1) If \( I \subseteq \mathbb{R}[x_0, \ldots, x_n] \) is a homogeneous tropical ideal, then \( \text{in}_w(I) \subseteq \mathbb{B}[x_0, \ldots, x_n] \) is a homogeneous tropical ideal, so each degree \( d \) part defines a (non-valuated) matroid \( \text{Mat}(\text{in}_w(I)) \).

(a) We define the \( w \) weight of a basis \( B \) of the degree \( d \) matroid to be \( \sum_{i \in B} w_i - \rho(B) \), where \( \rho \) is the basis valuation function for the valuated matroid of \( \text{Mat}(I_d) \). Show that the bases of \( \text{Mat}(\text{in}_w(I)_d) \) are the bases of \( \text{Mat}(I_d) \) of maximal weight.

(b) Show that the circuits of \( \text{Mat}(\text{in}_w(I)_d) \) are those initial terms of circuits of \( \text{Mat}(I_d) \) that have minimal support.

(2) (If you know what a matroid polytope is). Compare the matroid polytope of the underlying matroid of \( \text{Mat}(I_d) \) and the matroid polytope of \( \text{Mat}(\text{in}_w(I)_d) \). If you know what a regular subdivision is, how is that relevant here?

(3) Show that a tropical ideal is not necessarily determined by a finite set of degrees. Hint: Monday Exercise 4, supplementary 3.

(4) Tropical ideals obey the weak Nullstellensatz: \( V(I) = \emptyset \) if and only if \( I = \langle 0 \rangle \). Give an example to show that this is false for an arbitrary ideal in \( \mathbb{R}[x_1, \ldots, x_n] \).

(5) Let \( I \) be a non-homogeneous tropical ideal in \( \mathbb{R}[x] \). This means that for any finite collection \( E \) of monomials in \( \mathbb{R}[x] \), the set of polynomials in \( I \) supported in \( E \) is the set of vectors of a valuated matroid. Let \( f \) be a polynomial in \( I \) of lowest degree. Show that \( V(I) = V(f) \).