TUESDAY EXERCISE 4
BERKOVICH SPACES, PART I

Let K be an algebraically closed field that is topologically complete with respect to a non-trivial norm $|-|: K \longrightarrow \mathbb{R}_{\geq 0}$. Let $K[x]$ denote the polynomial ring in a single variable. The Zariski spectrum of $K[x]$ is the affine line $\mathbb{A}^1_K$. In this exercise, we explore the geometry of the Berkovich analytification $\mathbb{A}^{1,\mathrm{an}}_K$, often called the *Berkovich affine line*.

Let $f(x) = \sum_{n=0}^{d} a_n x^n \in K[x]$. For each $r \in \mathbb{R}_{>0}$ we define a map

$$|-|_{0,r} : K[x] \longrightarrow \mathbb{R}_{\geq 0}$$

$$|f|_{0,r} := \max\big\{|a_n| r^n\big\}_{n=0}^{d},$$

where the usual norm $|-|$ on K is being applied to each coefficient.

(1) Prove that $|-|_{0,r}$ is a seminorm on $K[x]$.

(2) ***Bonus*:** Prove that $|-|_{0,r}$ is multiplicative.

(3) Assuming that each seminorm $|-|_{0,r}$ is multiplicative, show that the family of seminorms $\{|-|_{0,r}\}_{r \in \mathbb{R}_{>0}}$ gives an inclusion of sets

$$\mathbb{R}_{>0} \hookrightarrow \mathbb{A}^{1,\mathrm{an}}_K.$$

(4) ***Bonus*:** Show that the map $\mathbb{R}_{>0} \hookrightarrow \mathbb{A}^{1,\mathrm{an}}_K$ realizes $\mathbb{R}_{>0}$ as a subspace of $\mathbb{A}^{1,\mathrm{an}}_K$.

(5) For each $a \in K$ and $r \in \mathbb{R}_{>0}$, define a map $|-|_{a,r} : K[x] \longrightarrow \mathbb{R}_{\geq 0}$ according to the following rule. Given a polynomial $f(x) \in K[x]$, let $\sum_{n=0}^{d} a_n (x-a)^n$ denote the expansion of $f(x)$ about a, and define

$$|f|_{a,r} := \max\big\{|a_n| r^n\big\}_{n=0}^{d}.$$

Prove that the seminorms $|-|_{0,r}$ and $|-|_{a,r}$ are distinct if $r < |a|$, and are identical if $r \geq |a|$. What does this imply about the geometry of $\mathbb{A}^{1,\mathrm{an}}_K$?