WEDNESDAY EXERCISE 1
TROPICAL IDEALS, PART 3

In this exercise we will look at three different ideals in $\mathbb{T}[x, y, z]$. There is (1) the principal ideal $\langle x + y + z\rangle$, (2) the Maclagan-Rincon tropical ideal, and (3) the tropical ideal $[x + y + z]$ constructed using stable sum.

(1) In degree 2, what are the elements in $[x+y+z]$ that are not in $\langle x+y+z\rangle$?

(2) Let MR_2 denote the (trivially valued) matroid of the Maclagan-Rincon ideal in degree 2.

  (a) What are the bases and co-bases (complements of bases)?

  (b) What are the circuits?

  (c) What are the co-circuits?

(3) Show that the Maclagan-Rincon tropical ideal in $\mathbb{T}[x, y, z]$ agrees with the tropical ideal $[x + y + z]$ in degrees 1 and 2.

(4) If you have time, try degree 3.

Supplementary exercises

(1) Given a vector $v \in \mathbb{T}^n$, we can think of this as representing a valuated matroid of rank 1 with basis valuation function $\rho_v(i) = v_i$. Given a second vector u, show that the basis valuation function for the stable sum $\rho_u \oplus^{st} \rho_v$ is given by sending $\{i, j\}$ to the corresponding permanent of the $2 \times n$ matrix with rows u and v.

(2) Show that the tropical hypersurfaces (i.e., tropical curves) in $\mathbb{P}^2_{\mathbb{T}}$ associated with the elements in the degree 2 part of $[x + y + z]$ all contain the tropical line defined by $x + y + z$.

(3) Now can you do this with $x + y + z$ replaced by an arbitrary homogeneous polynomials f of degree d?

(4) What is the matroid in degree 1 for the Maclagan-Rincon tropical ideal in $\mathbb{T}[x_0, x_1, x_2, x_3]$?

(5) Let $f = x^3 + x^2y + Cxy^2 + y^3 \in \mathbb{T}[x, y]$ for some nonvanishing $C \in \mathbb{T}$. Show that the tropical ideal $[f]$ is not the tropicalization of any ideal over a field of characteristic 2. *(Hint: Look at Plücker coordinates in degree 5.)*