WEDNESDAY EXERCISE 2

Recall that the ideal of a point $p \in \text{trop}(\mathbb{P}^n)$ is the ideal

$$I_p = \langle f : p \in V(f) \rangle.$$

(1) Show that the circuits of degree d are all $(p \cdot \mathbf{v}) \odot x^{\mathbf{u}} \oplus (p \cdot \mathbf{u}) \odot x^{\mathbf{v}}$. (You may assume for simplicity that all coordinates of p are finite).
(2) Conclude that I_p has Hilbert polynomial $P_I = 1$ (so has "degree one").
(3) Let p, q be two points in $\mathbb{P}^2$, and let J be the ideal of those two points in $K[x_0, x_1, x_2]$. Is $I_{\text{val}\, p} \cap I_{\text{val}\, q} = \text{trop}(J)$? (Hint: Think about the possibilities for the values, and valuations, of the coordinates of p, q).

Supplementary exercises

(1) Check that the addition on the hyperaddition on the three hyperfields discussed is associative. When A is a set, $A \boxplus b$ means $\cup_{a \in A} a \boxplus b$.

(2) For a hyperfield F, -1 is an element of F with $0_F \in 1 \boxplus -1$. What is this for the hyperfields you have seen? Is it unique?

(3) For a matroid over a hyperfield F, the basis valuation is a function $\rho \colon \binom{E}{d} \to F$ satisfying ρ is not the identically zero function, and, for all sets I of size $d-1$ and J of size $d+1$,

$$0_F \in \sum_{j \in J} (-1)^{\text{sign}} \rho(I \cup \{j\}) \rho(J \setminus \{j\}),$$

where sign is the standard sign for Plücker vectors (ask if you don't know it!). Show that for the Krasner hyperfield this defines usual matroids, and for the tropical hyperfield this defines valuated matroids.