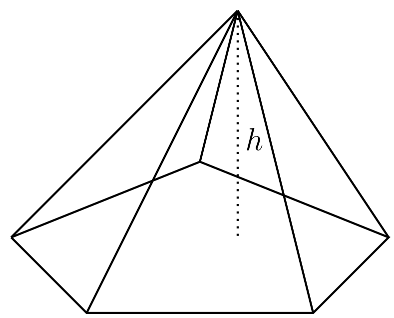
History and Applications of Calculus Homework 2

1. Prove that the volume of a pyramid with height h (measured perpendicular to the base) and base some shape of area A is $\frac{1}{3}Ah$:



Base area $= A$

2. This question is about the ellipse with foci F_1 and F_2 at coordinates $(c, 0)$ and $(-c, 0)$ in the xy-plane (for some given $c > 0$).

(a) Take $a > c$ and suppose that P is a point such that $|PF_1|+|PF_2| = 2a$, i.e., the distance from F_1 to P to F_2 is $2a$. The locus of all such points is an ellipse. Find the Cartesian equation of this ellipse in the standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Why is $b < a$?

(b) Use LaTeX to draw a sketch of this ellipse and label the axis intercepts too. The x-axis (where the two foci lie) is called the *major axis* and the y-axis is the *minor axis*. The number a is the length of the semi-major axis, and b is the length of the semi-minor axis.

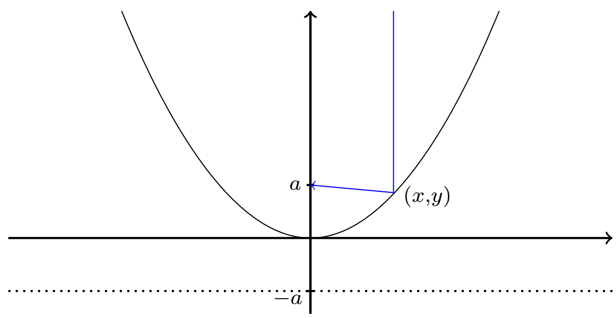
(c) The *eccentricity* $e < 1$ of the ellipse is defined by

$$e = \sqrt{1 - \frac{b^2}{a^2}}.$$

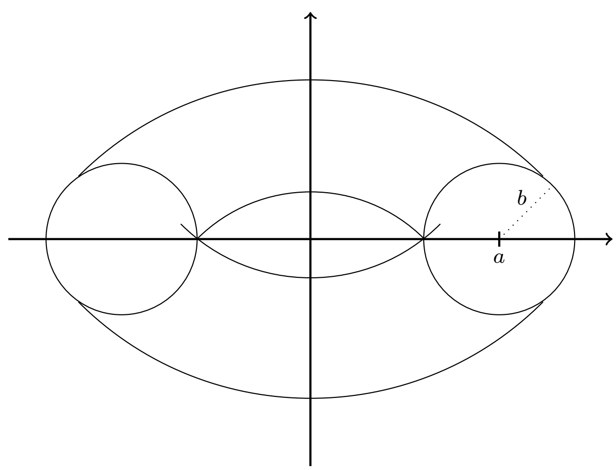
Calculate the x-coordinate c of the focus in terms of e and a and label the foci along with c on your graph.

3. Consider the parabola with equation $4ay = x^2$. Its *focus* is at the point $(0, a)$, its *directrix* is the line $y = -a$, and the parabola consists of all points

whose distance from the focus is equal to the distance from the directrix. If you make the parabola out of reflective material, i.e., it is a *parabolic mirror*, then you shine a ray of light vertically downwards onto the parabola, the parabolic mirror reflects all of the light to the focus. Prove this property!



4. Take two real numbers $a > b > 0$. Consider the circle centered at $(a, 0)$ of radius b. When you rotate this circle about the y-axis you generate a donut-shaped solid called a *torus*. Use the shell method to calculate the volume of this torus.



5. A classical theorem proved by Pappus gives a more general result than question 4. Consider some planar shape drawn to the right of the y-axis (for example, it might be the circle from question 3 but it could be any other shape). Suppose that the area of the shape is A and that the center-of-mass of the shape is distance a to the right of the y-axis. Prove that the volume of the solid that you get when you rotate the shape around the y-axis is equal to $2\pi aA$, i.e., it is the circumference of the circle of radius a times cross-sectional area A.