History and Applications of Calculus Homework 3

1. The cycloid is defined parametrically by the equations
   \[ x = a(t - \sin \theta), \quad y = a(1 - \cos \theta). \]
   Calculate the area under one “cycle” of this cycloid and above the \( x \)-axis.

2. The cardioid is defined in polar coordinates by the equation
   \[ r = 2a(1 - \cos \theta). \]
   Calculate the area of the cardioid.

3. Suppose \( F_1 \) and \( F_2 \) are two given foci in the plane which are a distance \( 2c \) apart, and \( 0 < a < c \). The hyperbola is the locus of points \( P \) such that \( \|PF_2| - |PF_1\| = 2a \).

   (a) Assuming that the foci are at coordinates \((\pm c, 0)\), show that the Cartesian equation for this hyperbola has the form
   \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]
   for some \( b > 0 \). Write \( b \) in terms of \( a \) and \( c \) and show that \( c = ae \) where \( e = \sqrt{1 + \frac{b^2}{a^2}} \) is the eccentricity.

   (b) Find the polar equation for this “left half” of this hyperbola, that is, the curve defined by the equation \( |PF_2| - |PF_1| = 2a \) (I’ve removed the absolute values!). Assume for this that \( F_1 \) is the origin and \( F_2 \) is the point \((2c, 0)\).

4. In this question you can assume without proof that the general solution of the differential equation
   \[ f''(x) + f(x) = 0 \]
   is \( f(x) = a \cos x + b \sin x \) where \( a = f(0) \) and \( b = f'(0) \), i.e., they are constants you can determine from the initial conditions. Use this to prove the multiple angle formulae
   \[ \sin(x + y) = \sin x \cos y + \cos x \sin y, \]
   \[ \cos(x + y) = \cos x \cos y - \sin x \sin y, \]
   \[ \tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}. \]
5. If you have a graph \( r = f(\theta) \) in polar coordinates, you can also consider the *inversion* of this graph, which is \( r = \frac{1}{f(\theta)} \). Note this transformation switches points inside the unit circle with points outside...

(a) The graphs \( r = 1 - \cos \theta \) and \( r = \frac{1}{1-\cos \theta} \) are the inversions of each other. Sketch these two graphs on the same axes. What are these two graphs called?

(b) Consider the straight line whose Cartesian equation is \( x = c \) for some constant \( c > 0 \). Show that its inversion is a circle passing through the origin.

(c) Show that the inversion of a circle which does *not* pass through the origin is another such circle.