History and Applications of Calculus Homework 4

1. A basic physics fact is that "work is force times distance." This means that to pull something along with constant force F Newtons for a total distance of x meters takes Fx Joules of energy... In this question, you'll also need Newton's Law of Gravity: the gravitational force between mass M and mass m which are a distance r apart is $F = GMm/r^2$ where G is the universal gravitational constant.

(a) The constant G is roughly $6.674 \times 10^{-11}\,\mathrm{m}^2\,/\,\mathrm{kg}\,.\,\mathrm{sec}^2$. Also the radius of the earth is about 4000 miles. What is the mass of the earth?

(b) Suppose you have a mass m distance r from another mass M. How much work do you need to do in order to move the mass m very very far away from the mass M ("to infinity")?

(c) In the lectures this week, we showed that the eccentricity of a planetary orbit around the sun satisfies the formula $e = \left(\frac{v_0}{v_{\text{crit}}}\right)^2 - 1$ where $v_{\text{crit}} = \sqrt{GM}/\sqrt{r_0}$, v_0 is the initial velocity of a planet of mass m moving in the tangential direction, r_0 is its initial distance from the sun, and M is the mass of the sun. How much kinetic energy does the planet have when $v_0 = \sqrt{2}v_{\text{crit}}$?

(d) Explain the connection between parts (b) and (c).

2. Recall that a function $f(x)$ with domain $\mathbb{R}$ is called *even* if $f(x) = f(-x)$ and *odd* if $f(x) = -f(-x)$ for all x.

(a) Suppose that $f(x)$ is *any* function with domain $\mathbb{R}$. Prove that there exist unique functions $E(x)$ and $O(x)$ such that $f(x) = E(x) + O(x)$ for every x, with $E(x)$ being even and $O(x)$ being odd. These functions $E(x)$ and $O(x)$ are called the *even* and *odd* parts of $f(x)$, respectively.

(b) Suppose that $f(x) = e^x$ is the exponential function. Sketch the graphs of $f(x)$, and its even and odd parts $E(x)$ and $O(x)$, all on the same axes.

3. The functions $E(x)$ and $O(x)$ in 2(b) are called the *hyperbolic functions* $\cosh x$ and $\sinh x$, respectively. Explicitly:

$$\cosh x = \frac{1}{2}(e^x + e^{-x}), \qquad \sinh x = \frac{1}{2}(e^x - e^{-x}).$$

(a) Show that the derivative of $\sinh x$ is $\cosh x$ and the derivative of $\cosh x$ is $\sinh x$.

(b) Show that $\cosh^2 x - \sinh^2 x = 1$.

(c) Let $\tanh x := \frac{\sinh x}{\cosh x}$ and $\operatorname{sech} x := \frac{1}{\cosh x}$. Prove that the derivative of $\tanh x$ is $\operatorname{sech}^2 x$.

4. Consider the differential equation

$$f''(x) - f(x) = 0.$$

(a) Prove that the functions $\cosh x$ and $\sinh x$ both give solutions of this differential equation, hence, so does any linear combination of the form $a \cosh x + b \sinh x$ for $a, b \in \mathbb{R}$. In fact, the latter is the *general solution* of this differential equation; you may assume this without proof in the remainder of the question.

(b) Prove that $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$. In particular, $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$.

(c) What is the analogous "double angle formula" for $\sinh 2x$?

5. In the lectures this week, we used laws of motion to derive the differential equation

$$\frac{d^2u}{d\theta^2} + u = \frac{f(r)r^2}{(v_0)^2(r_0)^2 m}$$

where $u = \frac{1}{r}$ and $f(r)$ was the central force, m was the mass of the planet, and v_0 and r_0 came from the initial conditions we were assuming (which implied $u = \frac{1}{r_0}$ and $\frac{du}{d\theta} = 0$ when $\theta = 0$). Then we assumed the central force was given by Newton's inverse square law $f(r) = \frac{GMm}{r^2}$, and solved this equation to obtain the final equation of motion; it was an ellipse, parabola or hyperbola according to the initial velocity v_0. I suggest you review this before attempting this question!

Taking the same general setup and initial conditions, suppose instead that we live in a parallel universe in which the central force satifies the *inverse cube law*

$$f(r) = \frac{GMm}{r^3}$$

for some alien constant G (it even has different units compared to our G!). What can you say about orbits of planets in this universe?