History and Applications of Calculus Homework 5

1.

(a) Use the substitution $x = \tanh u$ to calculate $\int \frac{dx}{1-x^2}$.

(b) Instead, calculate the integral in (b) by factoring $1-x^2 = (1+x)(1-x)$ then using the method of partial fractions. By comparing your answer to (a), deduce that

$$\operatorname{arctanh} x = \ln \sqrt{\frac{1+x}{1-x}}.$$

(c) Let $x = \tanh y = \frac{e^{2y}-1}{e^{2y}+1}$ (so $y = \operatorname{arctanh} x$). By solving the equation

$$x = \frac{e^{2y}-1}{e^{2y}+1}$$

for y, give another proof that $\operatorname{arctanh} x = \ln \sqrt{\frac{1+x}{1-x}}$.

(d) Find a formula for $\operatorname{arcsinh} x$ involving ln.

2. It may have been a while since you had to use "integration by parts," but you'll need it in this question! For $n = 1, 2, 3, \ldots$, let

$$I_n := \int_0^{\pi/2} \sin^n x \, dx.$$

(a) Use integration by parts to prove the following recurrence relation for the number I_n:

$$I_n = \frac{n-1}{n} I_{n-2}.$$

(b) Show that

$$\int_0^{\pi/2} \sin^{2n+1} \, dx = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \cdots \cdot \frac{2n}{2n+1},$$

$$\int_0^{\pi/2} \sin^{2n} x \, dx = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \cdots \cdot \frac{2n-1}{2n}.$$

Conclude that

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\dots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\frac{\int_0^{\pi/2}\sin^{2n}x\,dx}{\int_0^{\pi/2}\sin^{2n+1}x\,dx}.$$

(c) Show that the quotient of the two integrals in the expression just obtained is between 1 and $1+\frac{1}{2n}$. This shows that the products

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\dots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}$$

can be made as close to $\pi/2$ as desired by taking n sufficiently large. It is usually written as an infinite product known as Wallis' product:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\dots\,.$$

3. Now you'll need to remember how to calculate the Taylor series of a smooth (= infinitely differentiable) function $f(x)$ at $x=0$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots\,.$$

The series may or may not converge to the function $f(x)$, or it may only inverge inside some "radius of convergence," that is the point of Taylor's theorem. When it converges, it is perfectly legitimate to integrate or differentiate the function by integrating or differentiating the Taylor series "term by term" (it works for x inside the radius of convergence).

(a) The generalized binomial coefficient $\binom{\alpha}{n}$ is

$$\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$

which makes sense for any $\alpha\in\mathbb{R}$. Show that the Taylor series of the function $(1+x)^\alpha$ is

$$(1+x)^\alpha = \sum_{n\geq 0}\binom{\alpha}{n}x^n.$$

The radius of convergence of this series is 1 (this is a bit harder to prove and we won't do it here!).

(b) The Taylor series in (a) is called the *binomial series*. If α is actually a positive integer, show that $\binom{\alpha}{n} = 0$ for $n > \alpha$. In this case, the infinite sum in the binomial series collapses to just a finite sum, and the formula is called the *binomial theorem*, which should be familiar to you!

(c) If you replace x by $-x$ and take $\alpha = -1$ in the binomial series from (a), you get the *geometric series* for the function

$$\frac{1}{1-x}.$$

Work this series out explicitly using the formula from (a)!

(d) Now replace x by x^2 and take $\alpha = -1$ to get the Taylor series of the function $\frac{1}{1+x^2}$. Then integrate both sides to prove that

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

(The radius of convergence is 1 again.)

(e) Similarly derive the Taylor series of the function $\arcsin x$.

4. Here is another much more efficient way to calculate the number π.

(a) Show that

$$\arctan X + \arctan Y = \arctan\left(\frac{X+Y}{1-XY}\right).$$

(b) Use the identity from (a) to show that

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3},$$
$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

Using the second of these identities and some of the first few terms of the Taylor series for $\arctan x$, show that $\pi = 3.14159\cdots$. If you use a few more terms in the Taylor series, you can get lots of digits of π very quickly this way!

5. Calculate the Taylor series expansions of the functions $\tan x$ up to and including the x^7 term, and $\sec x$ up to and including the x^6 term.