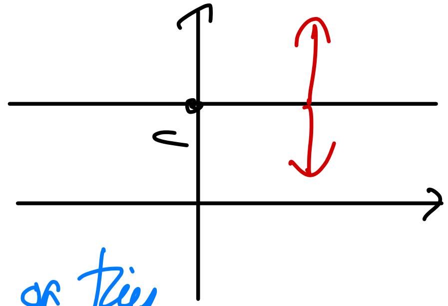
Differential equations ← An equation involving a derivative

(eg) $f'(x) = 0$

$\therefore \; f(x) = c$, constant ← Formal proof of this uses Mean Value Theorem

G.S. (general solution)



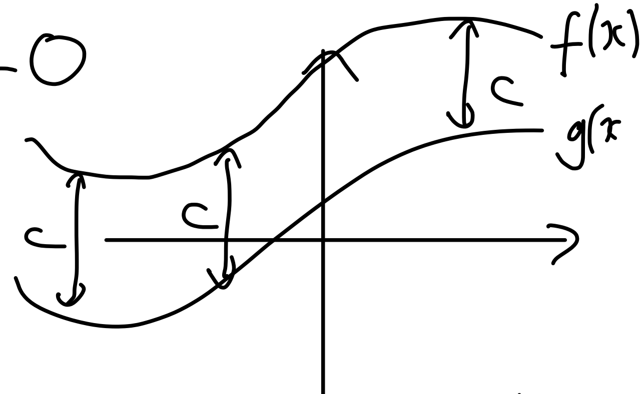
horizontal line, slope $= 0$ everywhere

(eg) If $f'(x) = g'(x)$, what can you say about f and g?

Let $h(x) = f(x) - g(x)$, so $h'(x) = f'(x) - g'(x) = 0$

$\therefore \; h(x) = c$, constant

$\therefore \; f(x) = g(x) + c$



f & g are "parallel" graphs

a.k.a. indefinite integral

Notation $\int f(x) \cdot dx$ shorthand for an antiderivative of $f(x)$, i.e. some particular function of x whose $\frac{d}{dx}$ is $f(x)$...

(eg) $\int x^2 \cdot dx = \frac{x^3}{3} + c$ ← c constant, I don't bother writing this!!!

only defined uniquely up to adding a constant.

$$\int x^n \cdot dx = \begin{cases} \dfrac{x^{n+1}}{n+1} & \text{if } n \neq -1 \\ \ln x & \text{if } n = -1 \end{cases}$$

($n \in \mathbb{R}$ any real number)

Inverse power rule

"add one to power, divide by the new power"

Can do a bit better: domain of $\ln x$ is $(0, \infty)$

Can also consider $\ln(-x)$ with domain $(-\infty, 0)$

$$\frac{d}{dx}\left(\ln(-x)\right) = \frac{1}{-x} \times (-1) = \frac{1}{x}$$

$$\int x^n \cdot dx = \begin{cases} \dfrac{x^{n+1}}{n+1} & \text{if } n \neq -1 \\ \ln |x| & \text{if } n = -1 \end{cases}$$

domain $\mathbb{R} - \{0\}$

Two examples of first order differential equations

k some given constant

① $\frac{dy}{dx} = kx$

$\therefore \int \frac{dy}{dx} \cdot dx = \int kx \cdot dx + c$

$\therefore \int dy = \int kx \cdot dx + c$

$\therefore y = \frac{1}{2}kx^2 + c$, c constant

G.S.

② $\frac{dy}{dx} = ky \iff \frac{1}{y}\frac{dy}{dx} = k$

$\therefore \int \frac{1}{y}\frac{dy}{dx} \cdot dx = \int k \cdot dx + c$

$\therefore \int \frac{1}{y} \cdot dy = \int k \cdot dx + c$, c constant

$\therefore \ln|y| = kx + c$

$\therefore |y| = e^{kx+c} = e^c e^{kx}$

$\therefore y = \underbrace{\pm e^c}_{A} e^{kx}$

$\therefore y = Ae^{kx}$

We just showed:-

$$\int u \frac{dy}{dx} \cdot dx = \int u \cdot dy + c$$

"substitution formula"

What's $\int x \cdot dx$? $\frac{x^2}{2}$ $\qquad \int y \cdot dy$? $\frac{y^2}{2}$

What's $\int u \frac{dy}{dx} \cdot dx$? ← u and y are both functions of x

Call it v, so $\frac{dv}{dx} = u\frac{dy}{dx}$. Times b.s. by $\frac{dx}{dy}$, get $\frac{dx}{dy}\frac{dv}{dx} = u\frac{dy}{dx}\frac{dx}{dy}$

$\therefore \frac{dv}{dy} = u$. This says that v is an anti-derivative of u wrt y. So: $v = \int u \cdot dy + c$

① $\frac{dy}{dx} = kx$

② $\frac{dy}{dx} = ky$

<u>Separate variables</u> !!! Move all y stuff to LHS, all x stuff to RHS

dy , dx

$$\therefore \int dy = \int kx \, dx + c$$

$$\therefore y = \frac{1}{2}kx^2 + c$$

G.S.

$$\therefore \int \frac{1}{y} dy = \int k \, dx + c$$

$$\ln|y| = kx + c$$

$\vdots$

$$y = Ae^{kx} \qquad A = \pm e^c$$

new constant

G.S.

Warning : not all first order diff. eq^n's work this way !!

Usually CANNOT separate variables.

eg $\quad y^2 + 2xy\frac{dy}{dx} + x^2 = 0$

Find G.S., then P.S. given that $y=0$ when $x=1$.

particular solution

What can you say about domain of function $y = y(x)$ in this P.S.?

Trick: $\frac{d}{dx}(xy^2) + x^2 = 0$ is our equation!!!!

$$\frac{d}{dx}(y^2) = \frac{dy}{dx}\cdot\frac{d}{dy}(y^2) = 2y\frac{dy}{dx}$$

$$\therefore \int d(xy^2) = -\int x^2\,dx + c$$

$$\therefore xy^2 = -\frac{x^3}{3} + c$$

$$\therefore y^2 = \frac{c}{x} - \frac{x^2}{3}, \quad c \text{ constant}$$

G.S.

Plug in $y=0$, $x=1$ ... $0 = c - \frac{1}{3}$ $\therefore c = \frac{1}{3}$

$$\therefore y^2 = \frac{1}{3}\left(\frac{1}{x} - x^2\right) \quad \therefore y = \sqrt{\frac{1}{3}\left(\frac{1}{x} - x^2\right)}$$

Domain $(0, 1]$

The above are first diff. eq's . . . . separation of variables.

There are second order ones, which involve second derivatives

$y = f(x) \quad \frac{dy}{dx} = f'(x) \quad \frac{d^2y}{dx^2} = f''(x) = (f')'(x)$

$$\frac{d}{dx}\left(\frac{d}{dx}(y)\right) = \frac{d^2y}{dx^2}$$

eg $\frac{d^2y}{dx^2} = k$, constant.

$\frac{dy}{dx} = kx + c$

$y = \frac{1}{2}kx^2 + cx + d$, c & d constants of integration

G.S.

(Need two initial conditions to find P.S.)