Finish discussion of differential equations

Often, \( y = y(t) \) is a function of time \( t \)

\[
\frac{dy}{dt} = \dot{y}(t) = \dot{y} \quad \text{"time derivative"}
\]

\[
\frac{d^2y}{dt^2} = \ddot{y}(t) = \ddot{y}
\]

If \( y \) is displacement of some moving particle
then \( \dot{y} \) is velocity and \( \ddot{y} \) is acceleration

"rate of change of displacement"  
"rate of change of velocity"

"time = \( \frac{\text{distance}}{\text{speed}} \)"
At time $t=0$, an apple falls from a tree. How long before it hits Sir Isaac on head?

Sir Isaac Newton

Gravity exerts a force on apple which accelerates it downwards. Near earth's surface, acceleration due to gravity is roughly constant, $g = 9.8 \text{ m/sec}^2$ ... force on apple is

$F = ma$

force = mass $\times$ accel.

mass of apple.

Let $y(t)$ = height of apple above Sir Isaac's head at time $t$, so $y(0) = 3$.

\[ y(t) = -gt + c \]

When $t=0$, $y(0) = 0$, so

\[ 0 = c \quad \text{so} \quad y = -gt \]

\[ y(t) = 3 - \frac{1}{2}gt^2 \]

When $t=0$, $y(0) = 3$, so $b = d$ ... so

\[ t = \sqrt{\frac{6}{g}} \approx 0.8 \text{ sec.} \]
So far: \( \frac{dy}{dx} \) derivatives (may not exist! need \( y \) to be differentiable)

\[ \int y \, dx \]
anti-derivatives / indefinite integral
(y may not have an anti-derivative, need \( y \) to be integrable)

Main topic today: the definite integral.

Whereas derivatives started from idea of slope, integrals start from idea of area

Know: area of rectangle: \( \text{area} = ab \)

area of triangle: \( \text{area} = \frac{1}{2} bh \)

But what about area for more wavy shapes?

Understood in elementary school ... count squarelets to estimate area.
Def. Let $f(x)$ be a function on domain $[a,b]$. Define the definite integral

$$\int_{a}^{b} f(x) \,dx = \text{area under the curve, above x-axis, in between } x=a \text{ and } x=b.$$ 

[area below x-axis counts as negative]

To calculate it, cut interval $[a,b]$ into thin strips of width $\delta x > 0$, then add up areas of all the strips.

Strip at $x$: $f(x)$

Approximately a rectangle of area $f(x) \cdot \delta x$

$$\int_{a}^{b} f(x) \,dx = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x) \cdot \delta x$$

$\delta x \to 0$ roughly means this limit exists.
The Fundamental Theorem of Calculus

Let \( f(x) \) be a continuous function on \([a, b]\).

Let

\[
F(x) = \int_a^x f(t) \, dt
\]

"area function"

Then:

\[
F'(x) = f(x) \quad \text{for all } x \in [a, b]
\]

i.e. \( F \) is an antiderivative of \( f \) on this interval.

(Shows in particular that all continuous functions are integrable.)
"Proof" \( f'(x) = \lim_{\delta x \to 0} \frac{F(x+\delta x) - F(x)}{\delta x} \)

\[ = \lim_{\delta x \to 0} \frac{\int_x^{x+\delta x} f(t) \cdot dt - \int_x^x f(t) \cdot dt}{\delta x} \]

\[ = \lim_{\delta x \to 0} \frac{\int_x^{x+\delta x} f(t) \cdot dt}{\delta x} \]

Connects definite integral (area) to indefinite integral (anti-derivative)

Corollary: If \( f(x) \) is continuous on \([a,b]\) and \( \int_a^b f(x) \cdot dx \) denotes some given anti-derivative, then

\[ \int_a^b f(x) \cdot dx = \left[ \int f(x) \cdot dx \right]^b_a \]

\[ \text{AREA} \]

Figurative proof here needs to use definition of continuous at this point. (Need to know continuous function on a closed banded interval is uniformly continuous)

\[ \text{Notation: } \left[ g(x) \right]^b_a \]

\[ \text{Means: } g(b) - g(a) \]
Proof of corollary: Let \( g(x) = \int f(x) \cdot dx \), the given antiderivative of \( f(x) \).

Let \( F(x) = \int_a^x f(t) \cdot dt \).

By FTC, \( F'(x) = f(x) = g'(x) \)

\( \Rightarrow F(x) = g(x) + C \), some constant \( C \).

\( \Rightarrow F(a) = \int_a^a f(t) \cdot dt = 0 = g(a) + C \)

What's \( C \)? Plug \( x = a \)...

\( \Rightarrow C = -g(a) \).

So \( F(x) = g(x) - g(a) \).

Now plug \( x = b \) to get

\( F(b) = \int_a^b f(x) \cdot dx = g(b) - g(a) = \left[ \int_a^b f(x) \cdot dx \right]_a^b \),

Note: Corollary holds ever without "continuous" assumption on \( f(x) \), merely need integrable.