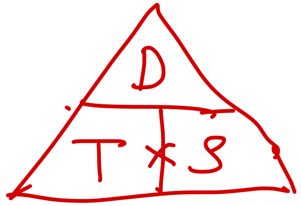
Finish discussion of differential equations

Often, $y = y(t)$ is a function of time t

$$\frac{dy}{dt} = \dot{y}(t) = \dot{y} \qquad \text{"time derivative"}$$

$$\frac{d^2y}{dt^2} = \ddot{y}(t) = \ddot{y}$$

"time = $\frac{\text{distance}}{\text{speed}}$"

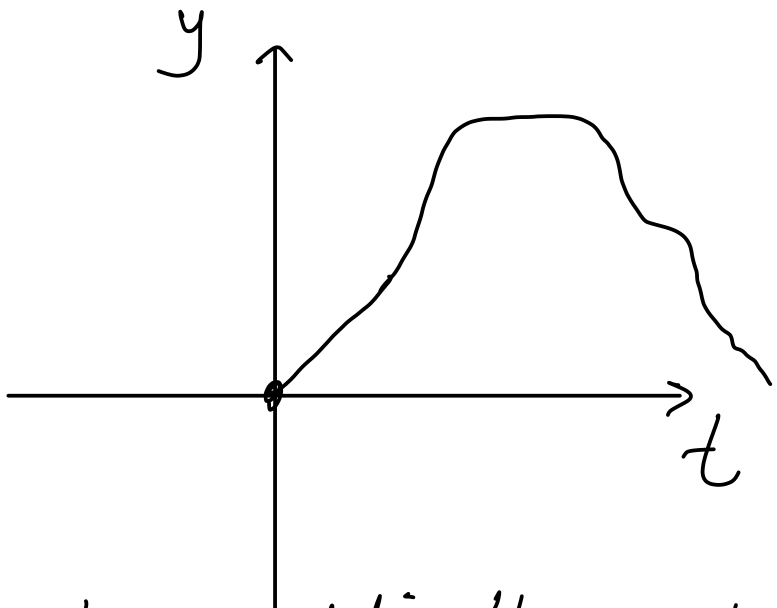


If y is displacement of some moving particle

then $\dot{y}$ is velocity and $\ddot{y}$ is acceleration

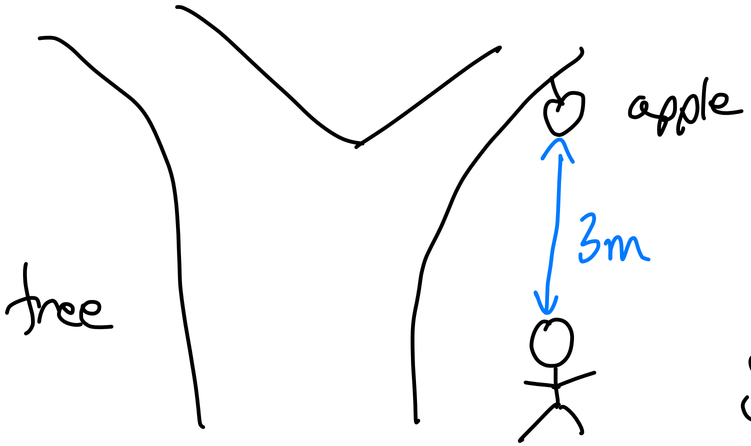
"rate of change of displacement"

"rate of change of velocity"



slope in a distance/time graph

is velocity



Sir Isaac Newton

At time $t=0$, apple falls from tree. How long before it hits Sir Isaac on head?

"$F = ma$"
force = mass × accel.

Gravity exerts a force on apple which accelerates it downwards.

Near earth's surface, acceleration due to gravity is roughly constant, $g = 9.8\ m/sec^2$ … force on apple is mg where m is mass of apple.

Let $y(t)$ = height of apple above Sir Isaac's head at time t, so $y(0) = 3$.

Physics $\Rightarrow$ $\boxed{\ddot{y} = -g}$ $\Longrightarrow$ $\dot{y} = -gt + c$

When $t=0$, $\dot{y}(0) = 0$, so $0 = c$ … so $\dot{y} = -gt$

Hence, $y = -\frac{1}{2}gt^2 + d$.

When $t=0$, $y(0) = 3$, so $3 = d$ … so $\boxed{y = 3 - \frac{1}{2}gt^2}$

Want t so $y(t) = 0$

$0 = 3 - \frac{1}{2}gt^2$

$\therefore t = \sqrt{\frac{6}{g}}$

≈ 0.8 sec.

So far: $\frac{dy}{dx}$ derivatives (may not exist! need y to be <u>differentiable</u>)

$\int y \cdot dx$ anti-derivatives / indefinite integral (y may not have an anti-derivative, need y to be <u>integrable</u>)

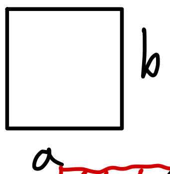
a function whose $\frac{d}{dx}$ is equal to y (if it exists, unique up to adding a scalar c)

<u>Main topic today</u>: the definite integral.

Whereas derivatives started from idea of slope, integrals start from idea of area

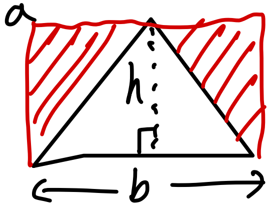
seemingly unrelated things!

<u>Know</u>: area of rectangle



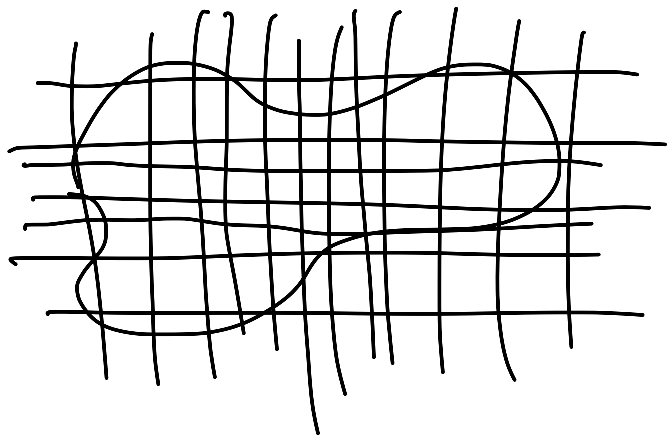
is ab

area of triangle



is $\frac{1}{2}bh$

The finer the grid the better!



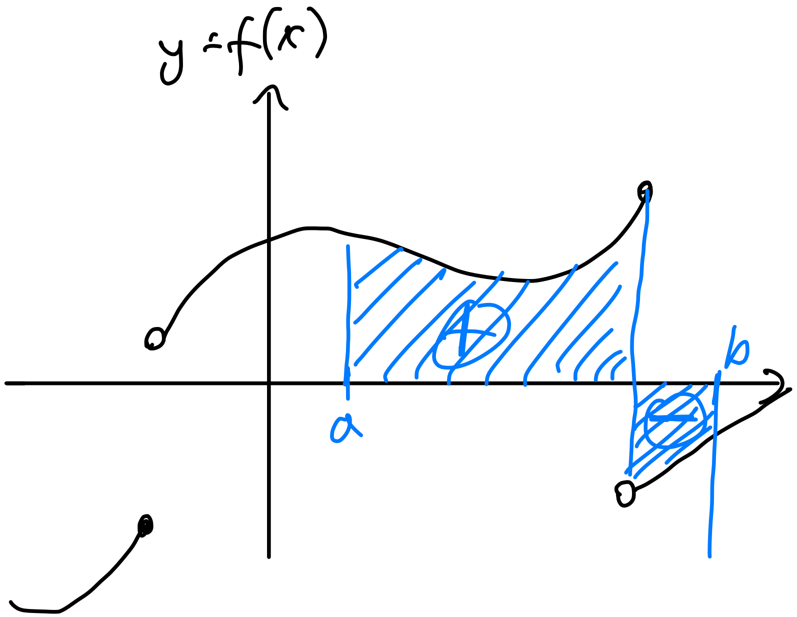
But what about area for more curvy shapes?

Understood in elementary school ... count squarelets to estimate area.

Def Let $f(x)$ be a integrable function on domain $[a,b]$. Define the definite integral

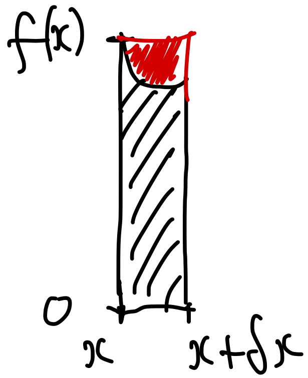
$$\int_a^b f(x)\cdot dx$$ = area under the curve, above x-axis, in between $x=a$ and $x=b$.

[area below x-axis counts as negative]



To calculate it, cut interval $[a,b]$ into thin strips of width $\delta x > 0$, then add up areas of all the strips.

Strip at x:



approximately a rectangle of area $f(x)\cdot\delta x$

$$\int_a^b f(x)\cdot dx = \text{"}\lim_{\delta x\to 0} \sum_{x=a}^{x=b} f(x)\cdot \delta x\text{"}$$

sum

"integrable" roughly means this limit exists.

The Fundamental Theorem of Calculus

Moment of enlightenment

Newton / Leibniz 1660/1670s

1665/1666

Newton was confined to his home village (plague years)

Let $f(x)$ be a continuous function on $[a,b]$.

Let

$$F(x) = \int_a^x f(t) . dt$$

"area function"

t dummy variable as x in use.

Then: $F'(x) = f(x)$ for all x in $[a,b]$

i.e. F is an anti-derivative of f on this interval.

(Shows in particular that all continuous functions are integrable)

"Proof" $F'(x) = \lim_{\delta x \to 0} \dfrac{F(x+\delta x) - F(x)}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{\int_a^{x+\delta x} f(t)\cdot dt - \int_a^x f(t)\cdot dt}{\delta x}$$

Connects definite integral (area) to indefinite integral (anti-derivative)

$$= \lim_{\delta x \to 0} \frac{\int_x^{x+\delta x} f(t)\cdot dt}{\delta x} \approx \lim_{\delta x \to 0} \frac{f(x)\cdot \cancel{\delta x}}{\cancel{\delta x}} = f(x)$$

Rigorous proof here needs to use definition of continuous at this point. (Need to know continuous function on a closed bounded interval is uniformly continuous)

Corollary If $f(x)$ is continuous on $[a,b]$ and $\int f(x)\cdot dx$ is some given anti-derivative, then

$$\int_a^b f(x)\cdot dx = \left[\int f(x)\cdot dx\right]_a^b$$

AREA

Notation $[g(x)]_a^b$ means $g(b) - g(a)$

Proof of corollary Let $g(x) = \int f(x) \cdot dx$, the given anti-derivative of $f(x)$

Let $F(x) = \int_a^x f(t) \cdot dt$.

FTC $\Rightarrow F'(x) = f(x) = g'(x)$

$\Rightarrow F(x) = g(x) + c$, some constant c.

What's c? Plug in $x = a$ ... $F(a) = \int_a^a f(t) \cdot dt = 0 = g(a) + c$

$\therefore c = -g(a)$.

So $F(x) = g(x) - g(a)$

Now plug in $x = b$ to get

$$F(b) = \int_a^b f(x) \cdot dx = g(b) - g(a) = \Big[\int f(x) \, dx \Big]_a^b$$

Note Corollary holds even without "continuous" assumption on $f(x)$; merely need integrable.