FTC Corollary

definite integral | indefinite integral

$$\int_a^b f(x)\cdot dx = \left[\int f(x)\cdot dx\right]_a^b$$

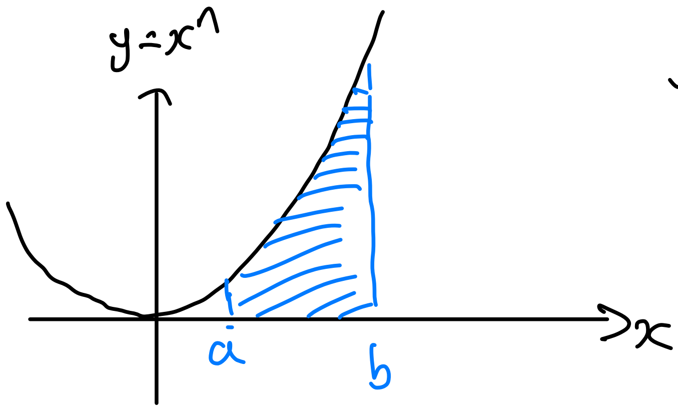
continuous

AREA under graph between $x=a$, $x=b$

ANY explicitly given anti-derivative of function $f(x)$.

- Idea of computing areas by cutting into strips was classical (Archimedes)
- Theorem of Fermat 1650-ish which clearly pointed to connection between (anti)-derivations and areas.

Fermat could calculate area under $y=x^n$ for $n\in\mathbb{N}$



$$\int_a^b x^n\cdot dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

Fermat showed $\int_0^b x^n\cdot dx = \frac{b^{n+1}}{n+1}$.

Quickly explain Fermat's argument...

Fermat's proof

Pre-requisite : geometric progressions / series

Let $s = \boxed{1+t+t^2+t^3+\cdots+t^n = \frac{1-t^{n+1}}{1-t}}$

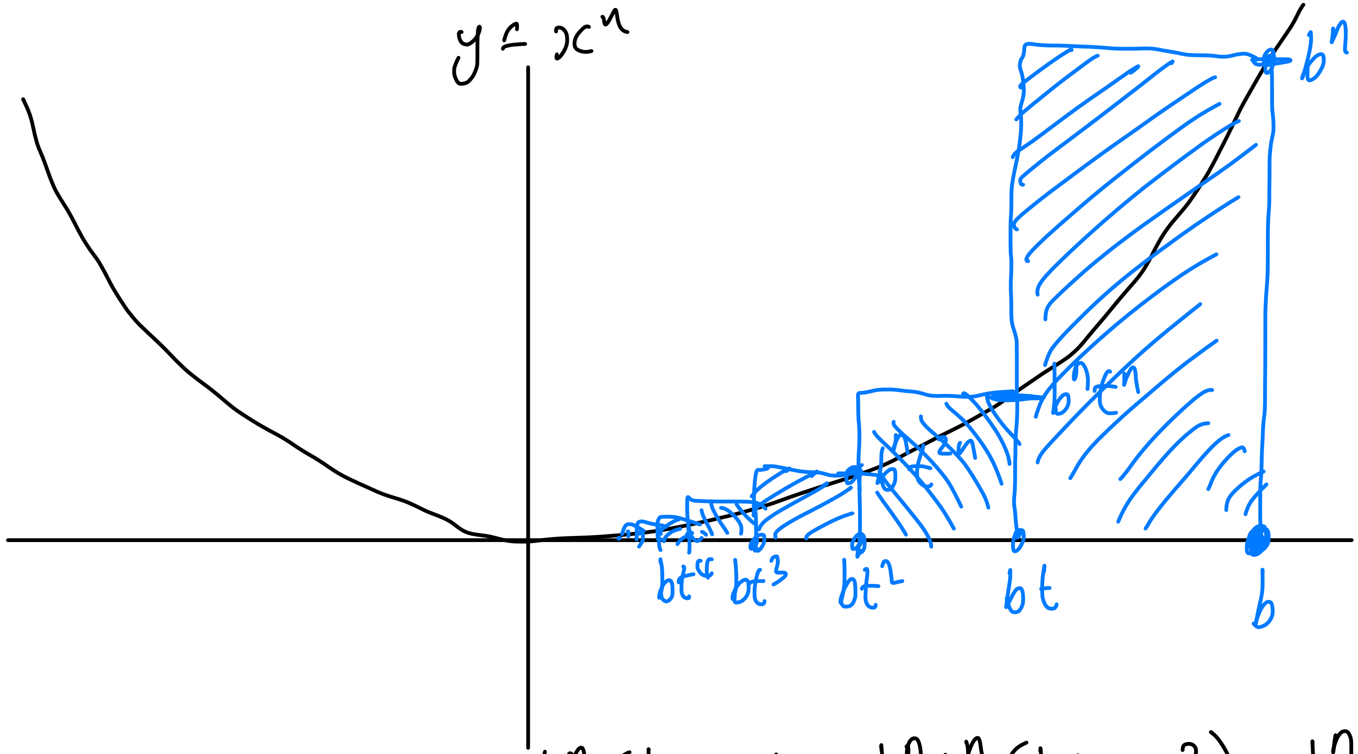
$(1-t)s = s-ts = 1+\cancel{t}+\cancel{t^2}+\cdots+\cancel{t^{n-1}}+\cancel{t^n}\cancel{-t}\cancel{-t^2}-t^3-\cdots\cancel{-}\cancel{-t^n}-t^{n+1} = 1-t^{n+1}$

$\therefore\ s = \frac{1-t^{n+1}}{1-t}$

Hence, if $|t|<1$, infinite sum $\boxed{1+t+t^2+t^3+\cdots = \frac{1}{1-t}}$

Proof LHS $\lim_{n\to\infty}\left(1+t+t^2+\cdots+t^n\right)$

$= \lim_{n\to\infty}\frac{1-t^{n+1}}{1-t} = \frac{1}{1-t}$



Take $0 < t < 1$

Consider strips at $x = b, bt, bt^2, bt^3, \dots$

Sum upper strips shown, then let $t \to 1^-$.

Area of upper strips $= b^n(b - bt) + b^n t^n (bt - bt^2) + b^n t^{2n}(bt^2 - bt^3) + \dots$

$$= b^{n+1}\left[(1-t) + t^n(t - t^2) + t^{2n}(t^2 - t^3) + \dots\right]$$

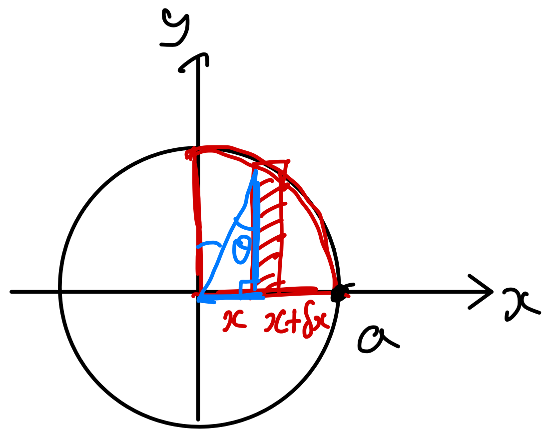
$$= b^{n+1}(1-t)\left[1 + t^{n+1} + t^{2(n+1)} + t^{3(n+3)} + \dots\right]$$

$$= b^{n+1}\frac{1-t}{1-t^{n+1}} = \frac{b^{n+1}}{1+t+t^2+\dots+t^n}.$$

$$\therefore \int_0^b x^n \, dx = \lim_{t\to 1^-}\left(\frac{b^{n+1}}{1+t+t^2+\dots+t^n}\right) = \frac{b^{n+1}}{n+1}$$

Examples of areas, volumes, arc lengths

① Area of circle of radius a



Graph is

$$x^2+y^2=a^2$$

$$y=\sqrt{a^2-x^2}$$

$$A=4\int_0^a \sqrt{a^2-x^2}\cdot dx$$

$$a^2-a^2\sin^2\theta$$
$$=a^2(1-\sin^2\theta)=a^2\cos^2\theta$$

" $\displaystyle\sum_{x=0}^{x=a}\sqrt{a^2-x^2}\cdot\delta x$ "

area of first quadrant

Let $x=a\sin\theta$

Then $\dfrac{dx}{d\theta}=a\cos\theta \qquad dx=a\cos\theta\cdot d\theta$

$$\leadsto \quad A=4\int_0^{\pi/2} a\cos\theta\cdot a\cos\theta\cdot d\theta$$

$$=4a^2\int_0^{\pi/2}\cos^2\theta\cdot d\theta$$

$$=4a^2\int_0^{\pi/2}\frac{\cos 2\theta+1}{2}\cdot d\theta$$

$$=2a^2\left[\tfrac{1}{2}\sin 2\theta+\theta\right]_0^{\pi/2}=2a^2\cdot\frac{\pi}{2}=\boxed{\pi a^2}$$

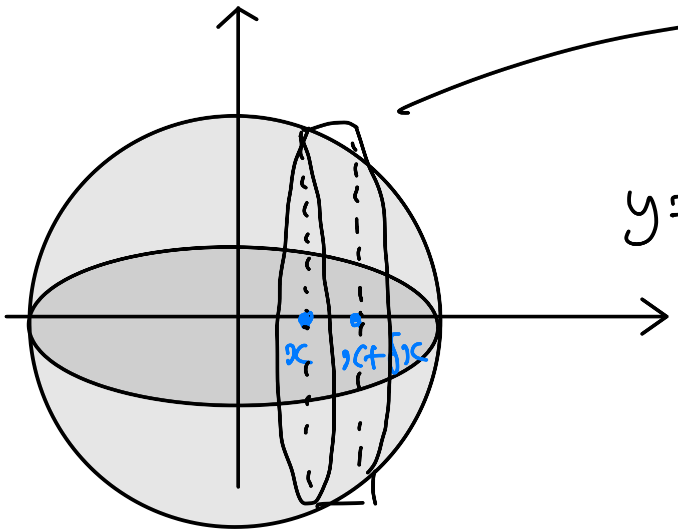
Double angle formulae

$$\sin 2\theta=2\sin\theta\cos\theta$$

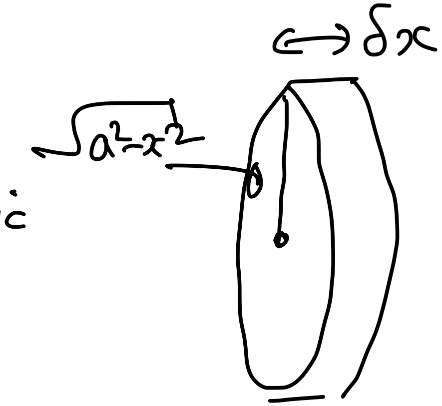
$$\left.\begin{aligned}\cos 2\theta&=\cos^2\theta-\sin^2\theta\\&=1-2\sin^2\theta\\&=2\cos^2\theta-1\end{aligned}\right\}$$

$$\therefore\ \cos^2\theta=\frac{\cos 2\theta+1}{2}$$

② Volume of sphere of radius a



thin disc



$y = \sqrt{a^2 - x^2}$

$$\text{Volume} = \pi(a^2 - x^2)\,\delta x$$

$$\text{"} \sum_{x=0}^{x=a} \pi(a^2 - x^2)\,\delta x \text{"}$$

$$V = 2\int_0^a \pi(a^2 - x^2)\cdot dx$$

$$= 2\pi\left[a^2 x - \frac{x^3}{3}\right]_0^a$$

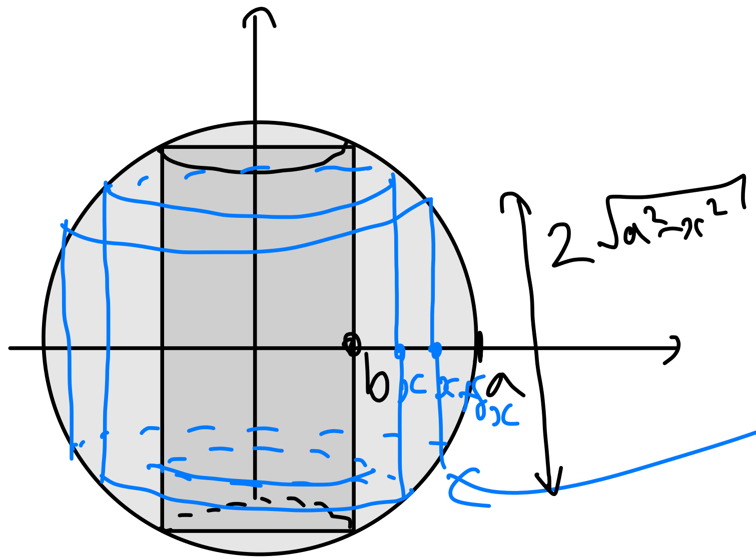
$$= 2\pi\left(a^3 - \frac{a^3}{3}\right) = 2\pi\cdot\frac{2a^3}{3} = \boxed{\frac{4}{3}\pi a^3}$$

DISC METHOD

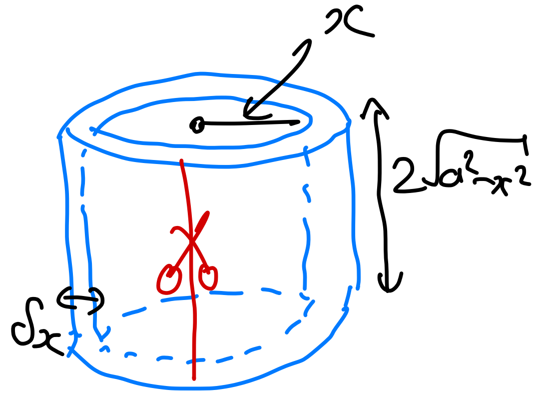
③ Volume of cored apple

Remove cylindrical core (cylinder radius b) from middle of spherical apple (radius a).

Volume of remaining piece of apple ?

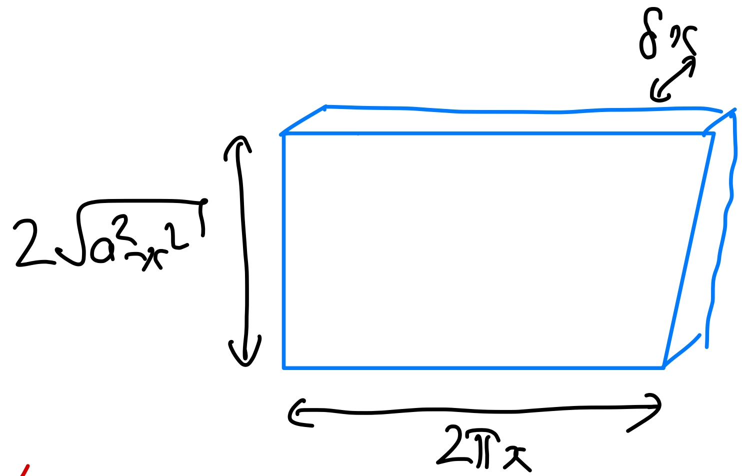


Cylindrical shell



Volume of shell

$$\text{“}\sum_{x=a}^{x=b} 4\pi x\sqrt{a^2-x^2}\,\delta x\text{”}$$



$$V = \int_b^a 4\pi x\sqrt{a^2-x^2}\,.\,dx$$

"friend"

$$= \left[-\frac{4\pi}{3}\left(a^2-x^2\right)^{3/2}\right]_b^a$$

$$= \boxed{\frac{4\pi}{3}\left(a^2-b^2\right)^{3/2}}$$

If b = 0 … volume of sphere ✓