Training phase \( \xrightarrow{\text{conic sections}} \) \( \xrightarrow{\text{parametric equations for curves}} \) \( \xrightarrow{\text{polar coordinates}} \) planetary motion

Parametric equations:
- Tangential slope: \(-\cot \theta\)
- Radial slope: \(\tan \theta\)

Distance:
- \(x = a \cos \theta\)
- \(y = a \sin \theta\)

\(\theta\) is a parameter, as \(\theta\) varies the point \((x, y)\) moves around the curve.

For circle:
- \(\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\)
- \(-\cot \theta = -\frac{1}{\tan \theta}\) shows tangent to circle is \(\theta\) to radius.
Arc length

\[ x = a \cos \theta \]
\[ y = a \sin \theta \]

Arc length for parametric curve from \( \theta = \theta_1 \) to \( \theta = \theta_2 \)

\[
\int_{\theta_1}^{\theta_2} \sqrt{\left( \frac{dx}{d\theta} \right)^2 + \left( \frac{dy}{d\theta} \right)^2} \, d\theta
\]

::: Circumference of circle of radius \( a \) = \[ 2\pi a \]

\[ \text{Arc length for sector of circle radius } a, \text{ angle } \theta \] = \[ 2\pi a \times \frac{\theta}{2\pi} = a\theta \]

Area of sector of circle radius \( a \), angle \( \theta \) = \[ \frac{1}{2} a^2 \theta \]
Cycloid

Take wheel of radius $a$, roll it along the x-axis.

Look at curve traced out by special part on circumference.

Length of one cycle of cycloid?

$8a$

Equation?

Let $\theta$: angle of rotation.

$$x = a\theta - a\sin\theta = a(\theta - \sin\theta)$$

$$y = a - a\cos\theta = a(1 - \cos\theta)$$

Parametric equations for cycloid.

$$\lim_{\theta \to \infty} \frac{dy}{dx} = \lim_{\theta \to \infty} \frac{\cos\theta}{\sin\theta} = +\infty$$
\[ x = a \left( \cos 0 + i \sin 0 \right) \]
\[ y = a \left( -\cos 0 \right) \]

Cycle length:

\[ \int_0^{2\pi} \sqrt{a^2(-\cos^2 0 + \sin^2 0)} \, d\theta \]
\[ = a \int_0^{2\pi} \sqrt{1 - 2\cos \theta + \cos^2 \theta} \, d\theta \]
\[ = a \int_0^{2\pi} \sqrt{2 - 2\cos \theta} \, d\theta \]
\[ = \sqrt{2} a \int_0^{2\pi} \sqrt{1 - \cos \theta} \, d\theta \]
\[ = \sqrt{2} a \int_0^{2\pi} \sqrt{2\sin^2 \frac{\theta}{2}} \, d\theta \]
\[ = 2a \int_0^{2\pi} \frac{1}{2} \sin \frac{\theta}{2} \, d\theta \]
\[ = 2a \left[ -\frac{2 \cos \frac{\theta}{2}}{2} \right]_0^{2\pi} = 4a \left( 1 + 1 \right) = 8a \]
Polar coordinates

\[ (x, y) \] in Cartesian coordinates
\[ (r, \theta) \] in polar coordinates

distance from 0 to the point
counterclockwise angle from x-axis

\[
\begin{align*}
    x &= r \cos \theta \\
y &= r \sin \theta
\end{align*}
\]

\[
r = \pm \sqrt{x^2 + y^2} \quad \theta = \arctan \left( \frac{y}{x} \right)
\]

The polar coordinates of a point are not unique!!!

eg \((r, \theta) = (1, \pi/4) = (1, 9\pi/4) = (-1, 5\pi/4) = \ldots\)

Standard convention: assume \( r \geq 0, \ -\pi < \theta \leq \pi \)

Alternative convention: all \( r \) (\( + \) or \( - \)), \( -\pi/2 < \theta \leq \pi/2 \)

\[ \arctan x \in (-\pi/2, \pi/2) \]
Graphs in polar coordinates

\[ r = f(\theta) \]

\[ y = f(x) \]

**Example:**

\( r = \theta \)

Not a function of angle

**Example:**

\( r = \cos \theta \)

\[ \theta = 0, \; r = 1 \]
\[ \theta = \frac{\pi}{2}, \; r = 0 \]
\[ \theta = \pi, \; r = -1 \]
\[ \theta = \frac{\pi}{4}, \; r = \frac{\sqrt{2}}{2} \]

Spiral

Circle radius \( \frac{1}{2} \)

Centered at \( x = \frac{1}{2}, \; y = 0 \)