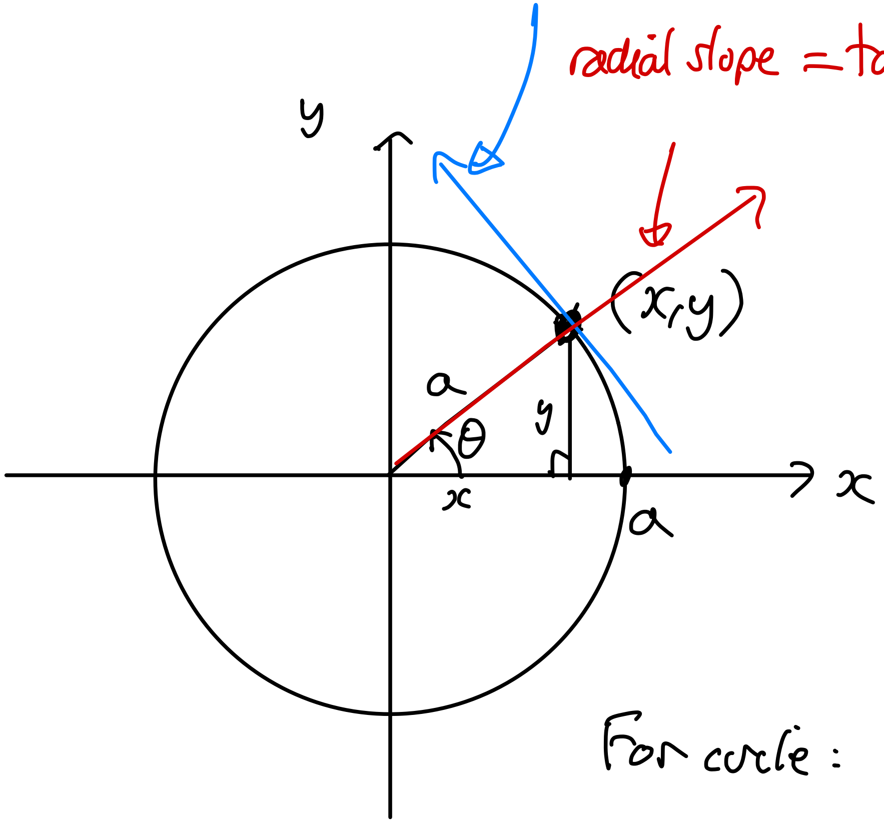
Training phase — conic sections, parametric equations for curves, polar coordinates $\Rightarrow$ planetary motion

Parametric equations

tangential slope $= -\cot\theta$

radial slope $= \tan\theta$



$$\left.\begin{aligned} x &= a\cos\theta \\ y &= a\sin\theta \end{aligned}\right\} \quad (x^2+y^2=a^2)$$

θ is a parameter, as θ varies the point (x,y) moves around the circle.

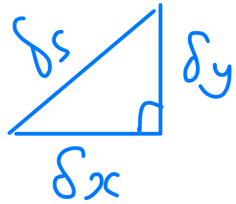
$$\boxed{\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}}$$

For circle: $\dfrac{dy}{dx} = \dfrac{\cancel{a}\cos\theta}{-\cancel{a}\sin\theta} = -\cot\theta$

$-\cot\theta = -\dfrac{1}{\tan\theta}$ $\rightsquigarrow$ shows tangent to circle is $\perp$ to radius.

Arc length

$$\left.\begin{aligned} x &= a\cos\theta \\ y &= a\sin\theta \end{aligned}\right\}$$



$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2$$

$$\therefore \delta s = \sqrt{(\delta x)^2 + (\delta y)^2}$$

$$\therefore \delta s = \sqrt{\left(\frac{\delta x}{\delta\theta}\right)^2 + \left(\frac{\delta y}{\delta\theta}\right)^2}\ \delta\theta$$

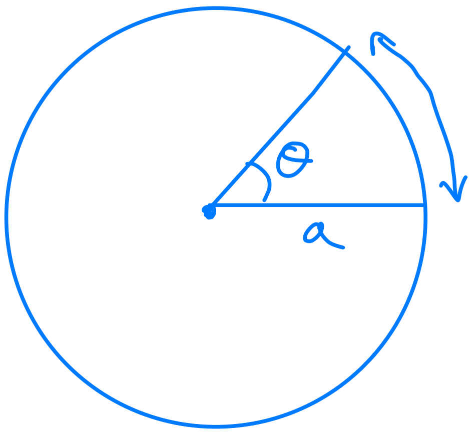
Arc length for parametric curve from $\theta = \theta_1$ to $\theta = \theta_2$

$$= \boxed{\int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\cdot d\theta}$$

" $\displaystyle\sum_{\theta=\theta_1}^{\theta_2} \delta s =$ "

$\therefore$ Circumference of circle of radius a $= \displaystyle\int_0^{2\pi} \sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\cdot d\theta = a\int_0^{2\pi} d\theta = \big[a\theta\big]_0^{2\pi}$

$$= \boxed{2\pi a}$$



Arc length for sector of circle radius a, angle θ

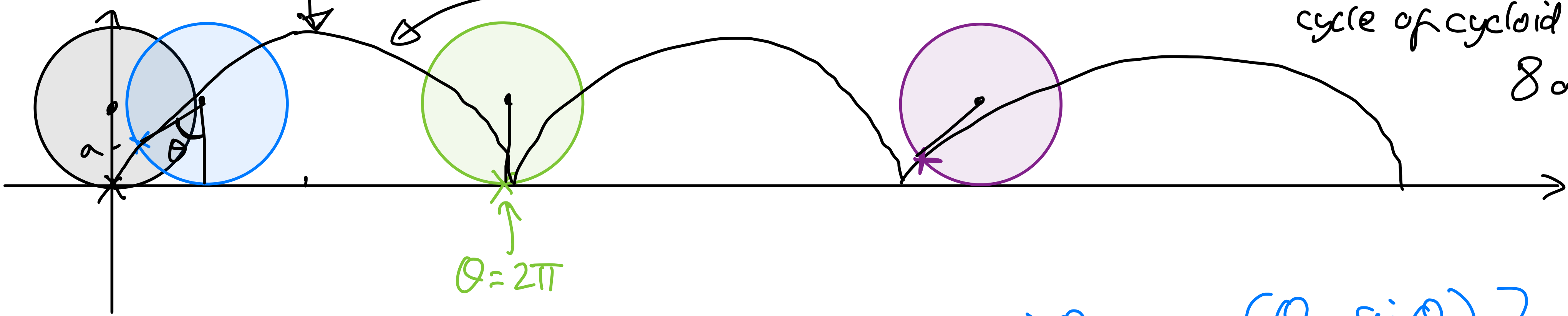
$$= \cancel{2\pi} a \times \frac{\theta}{\cancel{2\pi}} = \boxed{a\theta}$$

Area of sector of circle radius a, angle θ $= \cancel{\pi} a^2 \times \dfrac{\theta}{2\cancel{\pi}} = \boxed{\frac{1}{2}a^2\theta}$

Cycloid

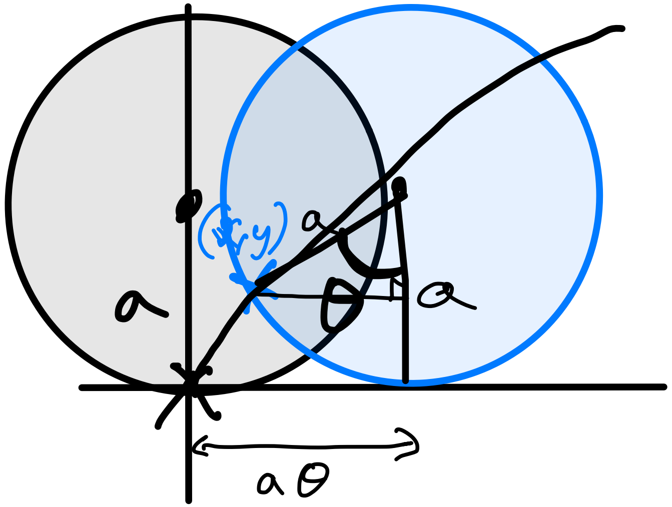
Take wheel of radius a, roll it along the x-axis.
Look at curve traced out by special point on circumference

length of one cycle of cycloid? $8a$



Equation?
Let θ = angle of rotation

$$x = a\theta - a\sin\theta = a(\theta - \sin\theta)$$
$$y = a - a\cos\theta = a(1 - \cos\theta)$$



$$\left.\begin{aligned} x &= a(\theta - \sin\theta) \\ y &= a(1 - \cos\theta) \end{aligned}\right\}$$ parametric equation for cycloid.

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{\cancel{a}\sin\theta}{\cancel{a}(1-\cos\theta)},$$

$$\lim_{\theta \to 0^+} \frac{dy}{dx} = \lim_{\theta \to 0^+} \frac{\cos\theta}{\sin\theta}$$
$$= +\infty$$

$$\left.\begin{aligned} x &= a(\theta - \sin\theta) \\ y &= a(1-\cos\theta) \end{aligned}\right\}$$

$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$

by double angle formula!

Cycle length $= \displaystyle\int_0^{2\pi} \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\; d\theta$

$$= a\int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\cdot d\theta$$

$$= a\int_0^{2\pi}\sqrt{2-2\cos\theta}\cdot d\theta = \sqrt{2}\,a\int_0^{2\pi}\sqrt{1-\cos\theta}\; d\theta$$

$$= \sqrt{2}\,a\int_0^{2\pi}\sqrt{2\sin^2\frac{\theta}{2}}\cdot d\theta = 2a\int_0^{2\pi}\sin\frac{\theta}{2}\cdot d\theta$$

$$= 2a\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi} = 4a(1+1) = \boxed{8a}$$

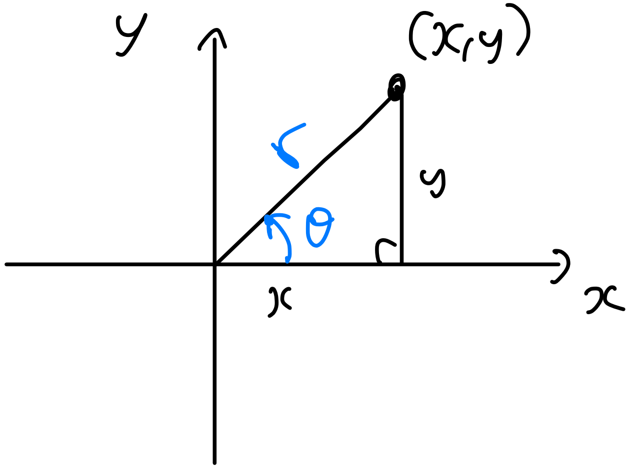
Polar coordinates

(x,y) in Cartesian coordinate

(r,θ) in polar coordinates

distance from O to the point

counterclockwise angle from x-axis



$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \qquad \left.\begin{aligned} r &= \pm\sqrt{x^2+y^2} \\ \theta &= \arctan\left(y/x\right) \end{aligned}\right\}$$

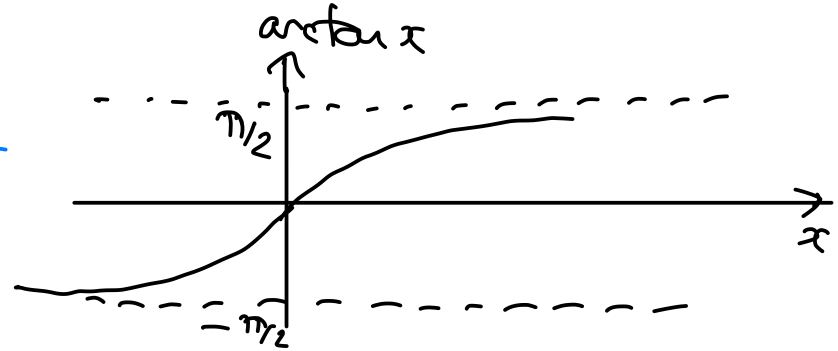
The polar coordinates of a point are not unique !!!

eg $(r,\theta) = (1, \pi/4) = (1, 9\pi/4) = (-1, 5\pi/4) = \cdots$

Standard convention : assume $r \geq 0$, $-\pi < \theta \leq \pi$

Alternative convention : all r (+ or −), $-\pi/2 < \theta \leq \pi/2$

$\arctan x \in (-\pi/2, \pi/2)$

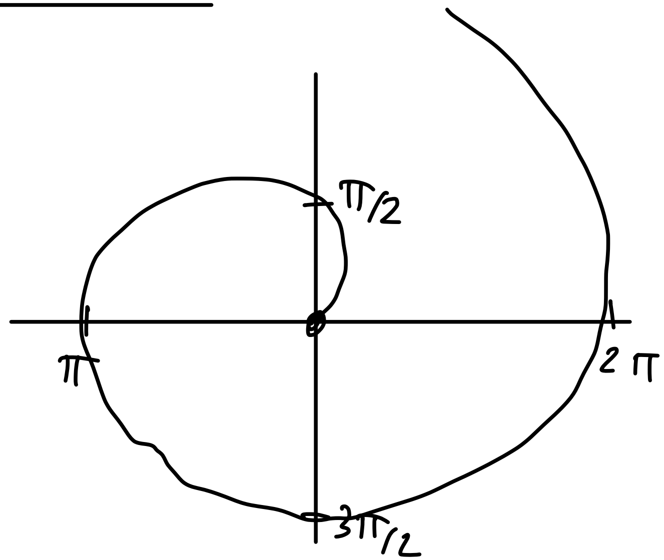


Graphs in polar coordinates

$r = f(\theta)$ $\quad$ $y \cancel{= f}(x)$

eg $r = \theta$

Not a function of angle



Spiral

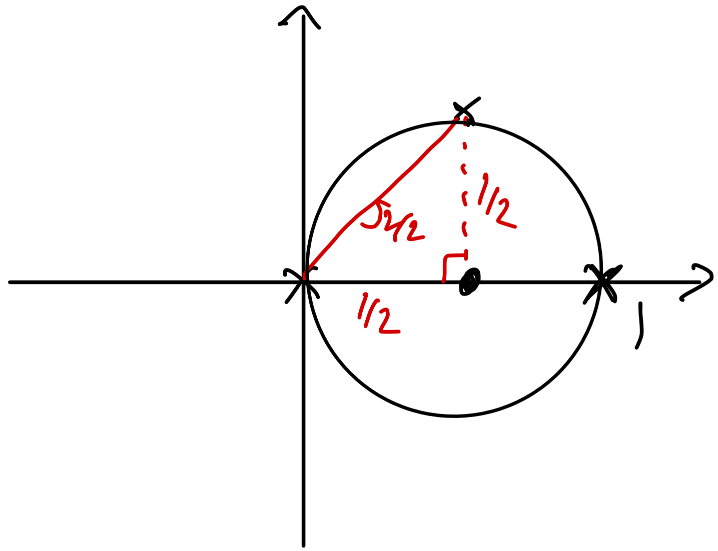
eg $r = \cos\theta$

$\theta = 0, \; r = 1$

$\theta = \pi/2, \; r = 0$

$\theta = \pi, \; r = -1$

$\theta = \pi/4, \; r = \sqrt{2}/2$



Circle radius $\frac{1}{2}$

Centered at $x = \frac{1}{2}, \; y = 0$