Finding areas in polar coordinates

\[ y = f(x) \quad r = f(\theta) \]

SO-sector has area = \( \frac{1}{2} f(\theta)^2 \cdot \delta \theta \)

\[ \sum_{\theta = \theta_1}^{\theta_2} \frac{1}{2} f(\theta)^2 \cdot \delta \theta \]

Area of sector under polar graph from \( \theta = \theta_1 \) to \( \theta = \theta_2 \) = \( \frac{1}{2} \int_{\theta_1}^{\theta_2} f(\theta)^2 \cdot d\theta \)

**Example**

\( r = \cos \theta \)

\[ A = \frac{1}{2} \int_{0}^{\pi} \cos^2 \theta \cdot d\theta = \frac{1}{2} \int_{0}^{\pi} \frac{\cos 2\theta + 1}{2} \cdot d\theta \]

\[ \pi \times (\frac{1}{2})^2 = \frac{\pi}{4} \]

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\[ \cos 2\theta = \frac{2 \cos^2 \theta - 1}{2} \]

\[ \frac{\pi}{4} = \frac{\pi}{4} \quad \checkmark \]
Cardioid

Roll red curve S around fixed grey curve

Look at curve traced out by special point

on circumference of rolling curve

Find its equation in polar coordinates.

\[
\sin \frac{\theta}{2} = \frac{b}{2a} = \frac{r}{2b} \quad \therefore \quad b^2 = ar
\]

\[
\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} = 1 - 2 \left( \frac{b}{2a} \right)^2
\]

\[
\therefore \quad \cos \theta = 1 - \frac{b^2}{2a^2} = 1 - \frac{a^2 r}{2a^2} = 1 - \frac{r}{2a}
\]

\[
\therefore \quad \frac{r}{2a} = 1 - \cos \theta
\]

\[
\therefore \quad r = 2a (1 - \cos \theta)
\]

HW: Find area of cardioid.
Back to conic sections.

**Ellipse**

- $a \geq b$
- $0 \leq e < 1$

$c = ae$ where $e = \sqrt{1 - \frac{b^2}{a^2}}$

\[\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \iff |PF_1| + |PF_2| = 2a\]

Where are the foci? $a^2 = b^2 + c^2$

$\therefore c = \sqrt{a^2 - b^2} = a \sqrt{1 - \frac{b^2}{a^2}}$ (eccentricity)

**Hyperbola**

- $e > 1$

\[\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \iff \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \iff y = \pm \frac{b}{a}x\]

Where are the foci? $c = ae$ where $e = \sqrt{1 + \frac{b^2}{a^2}}$
Similar triangles:
\[ \frac{\sqrt{a^2+b^2}}{a} = \frac{c}{a} \]

\[ \therefore \sqrt{a^2+b^2} = c \]

\[ \therefore c = a \sqrt{1 + \frac{b^2}{a^2}} \]

e, eccentricity
Parabola (Conic section of eccentricity $e=1$)

$4ay = x^2$

$|PF_1| = \sqrt{x^2 + (a-y)^2} = \sqrt{4ay + a^2 - 2ay + y^2}$

$= \sqrt{y^2 + 2ay + a^2} = \sqrt{(y+a)^2} = y + a = |PF_2|$

$\therefore |PF_1| = |PF_2|$ where $F_1 = (0, a)$ focus and $F_2$ is projection of $P$ onto directrix $y = -a$