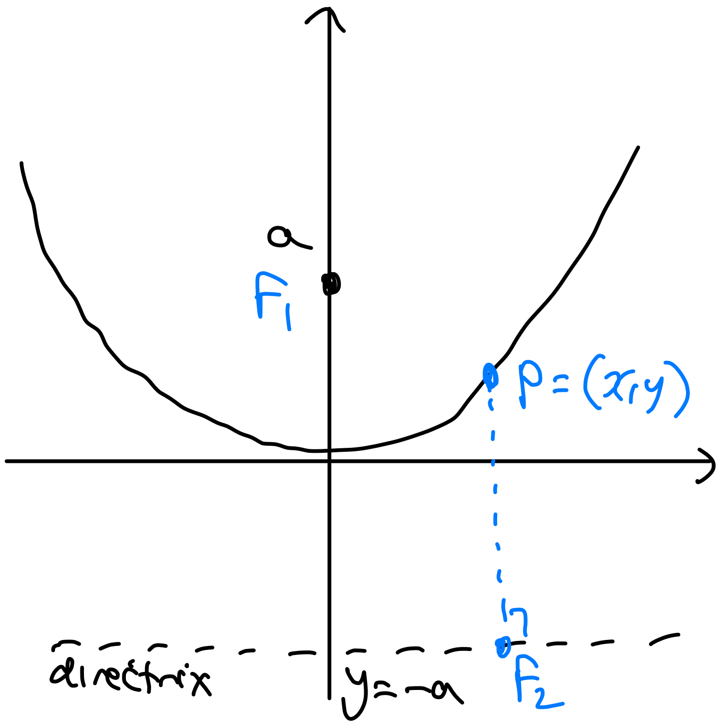
More about parabolas

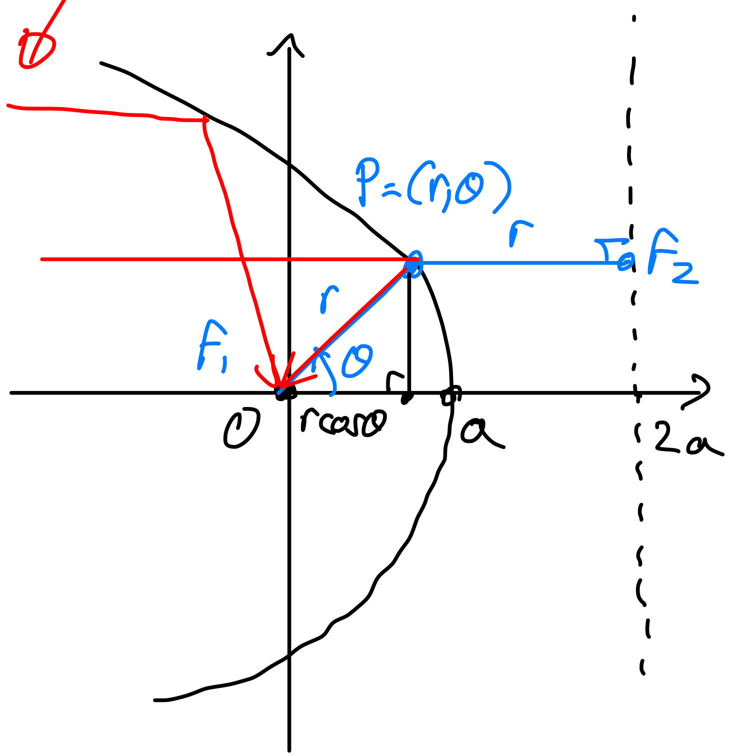


$$4ay = x^2 \iff |PF_1| = |PF_2|$$

Polar coordinates equation!

Goal: show that parabola is only curve with this optical property.

optical property of the parabola



$$r = 2a - r\cos\theta$$

$$\therefore\ r + r\cos\theta = 2a$$

$$\therefore\ r(1+\cos\theta) = 2a$$

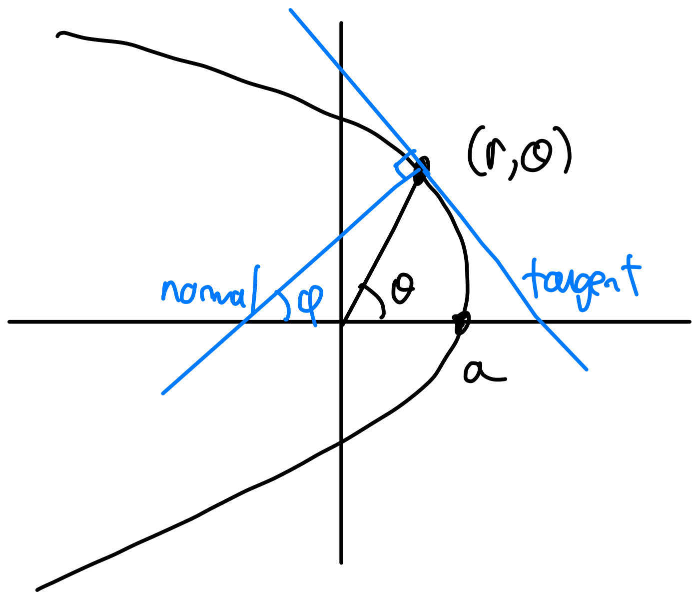
$$\therefore\ \boxed{r = \frac{2a}{1+\cos\theta}}$$

Polar equation for parabola.

$\boxed{e=1}$

$$r = \frac{r_0(1+e)}{1+e\cos\theta}$$

$r = f(\theta)$ some polar graph,



Let φ be the angle between the x-axis and the normal.

θ, φ ... what the connection?

First, what's slope of tangent line? $\frac{dy}{dx}$

$$\left.\begin{aligned} x &= f(\theta)\cos\theta \\ y &= f(\theta)\sin\theta \end{aligned}\right\} \quad \left.\begin{aligned} \frac{dx}{d\theta} &= f'(\theta)\cos\theta - f(\theta)\sin\theta \\ \frac{dy}{d\theta} &= f'(\theta)\sin\theta + f(\theta)\cos\theta \end{aligned}\right\}$$

slope of tangent line $\quad \frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{f'(\theta)\tan\theta + f(\theta)}{f'(\theta) - f(\theta)\tan\theta}$

slope of normal $\quad -\frac{1}{\frac{dy}{dx}} = -\frac{dx}{dy} = \frac{f(\theta)\tan\theta - f'(\theta)}{f'(\theta)\tan\theta + f(\theta)} = \tan\varphi$

$\therefore f(\theta)\tan\theta - f'(\theta) = f'(\theta)\tan\theta\tan\varphi + f(\theta)\tan\varphi$

$\therefore f(\theta)[\tan\theta - \tan\varphi] = f'(\theta)[1 + \tan\theta\tan\varphi] \Rightarrow$

$$\boxed{\frac{f'(\theta)}{f(\theta)} = \frac{\tan\theta - \tan\varphi}{1 + \tan\theta\tan\varphi}}$$

$$\boxed{\frac{f'(\theta)}{f(\theta)} = \frac{\tan\theta - \tan\varphi}{1+\tan\theta\tan\varphi}}$$

Relates angle θ of our point (r,θ) on the curve to angle φ of normal to horizontal

RHS is very nice...

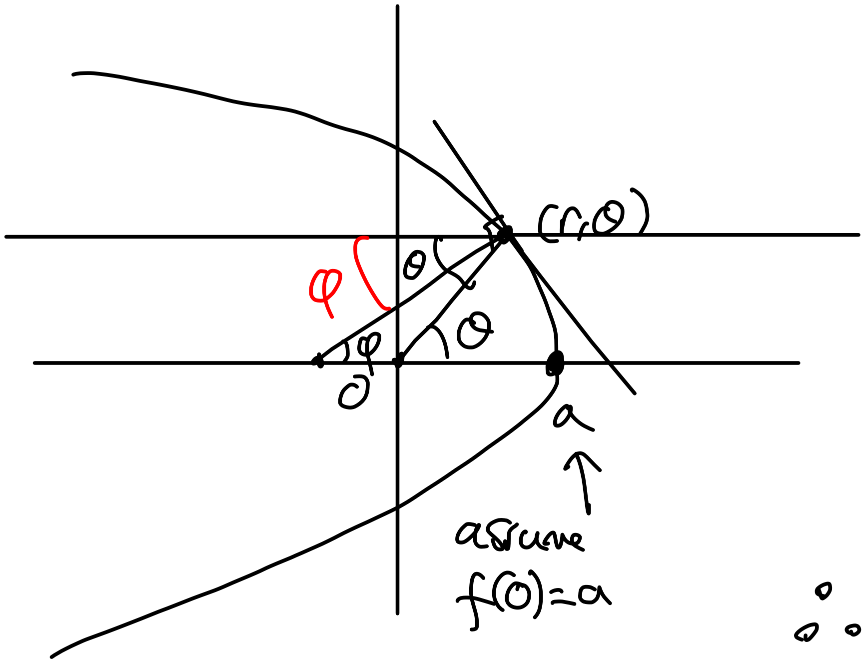
$$\left.\begin{aligned} \sin(\theta-\varphi) &= \sin\theta\cos\varphi - \cos\theta\sin\varphi \\ \cos(\theta-\varphi) &= \cos\theta\cos\varphi + \sin\theta\sin\varphi \end{aligned}\right\} \text{multiple angle formulae}$$

$$\therefore \tan(\theta-\varphi) = \frac{\sin\theta\cos\varphi - \cos\theta\sin\varphi}{\cos\theta\cos\varphi + \sin\theta\sin\varphi} = \frac{\tan\theta - \tan\varphi}{1+\tan\theta\tan\varphi}$$

$$\Rightarrow \boxed{\frac{f'(\theta)}{f(\theta)} = \tan(\theta-\varphi)}$$

Now assume polar graph $r = f(\theta)$ has the optical property

$$\boxed{\frac{f'(\theta)}{f(\theta)} = \tan(\theta - \varphi)}$$



Optical property : the normal bisects the angle θ.

$$\Updownarrow$$

$$\varphi = \frac{\theta}{2}$$

$$\therefore \frac{f'(\theta)}{f(\theta)} = \tan\left(\frac{\theta}{2}\right)$$

$$r = f(\theta)$$

$$\frac{dr}{d\theta} = f'(\theta)$$

$$\boxed{\therefore \frac{1}{r}\frac{dr}{d\theta} = \tan\frac{\theta}{2}}$$

first order differential equation for $r = f(\theta)$

Now we must solve this diff. eq!

$$\boxed{\therefore \frac{1}{r}\frac{dr}{d\theta} = \tan\frac{\theta}{2}}$$

$$\int \tan\frac{\theta}{2}\cdot d\theta = \int \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\cdot d\theta$$

$$= -2\ln\left(\cos\frac{\theta}{2}\right)$$

Separate variable

$$\int \frac{1}{r}\,dr = \int \tan\frac{\theta}{2}\,d\theta + c$$

$$\therefore \ln(r) = \ln\left(\frac{1}{\cos^2\frac{\theta}{2}}\right) + c \qquad A = e^c$$

$$\therefore r = \frac{A}{\cos^2\frac{\theta}{2}}$$ when $\theta = 0$, $r = a$ $\therefore A = a$

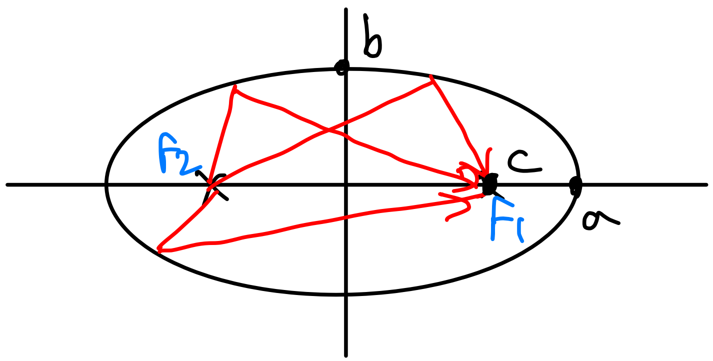
$$\therefore r = \frac{2a}{2\cos^2\frac{\theta}{2} - 1 + 1} = \frac{2a}{\cos\theta + 1}$$

$$\boxed{r = \frac{2a}{1+\cos\theta}}$$

Same as polar equation for parabola we derived at start of lecture!!!

What about ellipse?

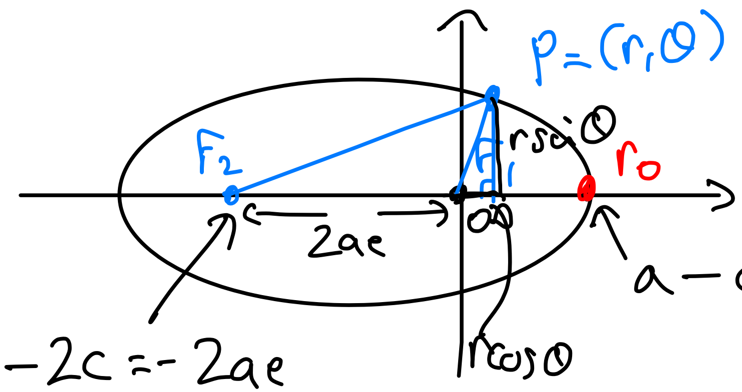
eccentricity $0 \le e < 1$



Also has optical property...

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad c = ae \qquad e = \sqrt{1 - b^2/a^2}$$

What about polar equation of ellipse? Move the origin to F_1



r_0

$$a - c = a - ae = a(1-e)$$

$$|PF_1| + |PF_2| = 2a$$

$$\therefore r + |PF_2| = 2a$$

$$|PF_2|^2 = (2a-r)^2$$

$$\therefore r^2\sin^2\theta + (r\cos\theta + 2ae)^2 = (2a-r)^2$$

$$\therefore \underbrace{r^2\sin^2\theta + r^2\cos^2\theta}_{\cancel{r^2}} + 4aer\cos\theta + 4a^2e^2 = 4a^2 - 4ar + \cancel{r^2}$$

$$4aer\cos\theta + 4a^2e^2 = 4a^2 - 4ar$$

$$er\cos\theta + ae^2 = a - r$$

$$\therefore r + re\cos\theta = a - ae^2$$

$$\therefore r(1+e\cos\theta) = \underbrace{a(1-e)}_{\text{Call this } r_0}(1+e)$$

$$\therefore \boxed{r = \frac{r_0(1+e)}{1+e\cos\theta}}$$

Unified polar equation for all conic sections.

e eccentricity

r_0 x-axis intercept

Polar equation for ellipse

Equation for parabola from earlier has same form $(e=1, a=r_0)$

Equation for hyperbola is also exactly this $(e>1)$.