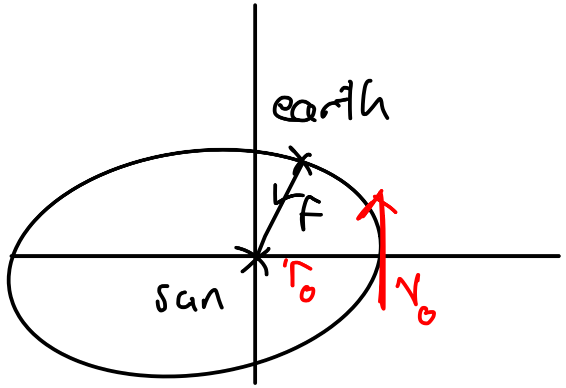
So far we've seen KII (equal areas in equal times) is equivalent to motion under central force.



Already shown:

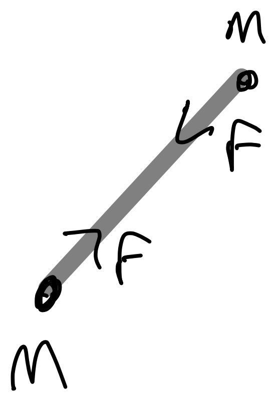
① $\underline{\ddot{x}} = \underbrace{(\ddot{r} - r(\dot{\theta})^2)}_{\text{radial acceleration}} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

② $r^2\dot{\theta} = r_0 v_0$, constant over all time.

Can use this to eliminate $\dot{\theta}$

Need <u>Newton's Law of Gravity</u>

$$F = \frac{GMm}{r^2}$$ force due to gravity



G = universal gravitational constant

Two masses M, m

distance r apart

Now we "f=ma" to get

$r^2\dot{\theta} = r_0 v_0$

$$-F = m(\ddot{r} - r(\dot{\theta})^2)$$

$$\therefore -\frac{GM\cancel{m}}{r^2} = \cancel{m}(\ddot{r} - r(\dot{\theta})^2)$$

$$\therefore GM = -r^2\ddot{r} + \frac{r_0^2 v_0^2}{r}$$

A bit nasty-looking! Also want r in terms of θ not in terms of t.

Helpful to think what are we expecting?

Ellipse ... $r = \dfrac{r_0(1+e)}{1+e\cos\theta}$ $\quad 0 \leq e < 1$

What diff. eq. could produce such a formula?

Suggest $u = \frac{1}{r}$ will work better

$u = a\cos\theta + c$ ← This I do know a diff. eq. which has this as its solution.

$$\frac{d^2u}{d\theta^2} + u = 0 \quad \text{has G.S.} \quad u = a\cos\theta + b\sin\theta$$

$$\frac{d^2u}{d\theta^2} + u = c \quad \text{has G.S.} \quad u = \underbrace{a\cos\theta + b\sin\theta}_{\text{G.S. of associated homogeneous equation}} + \underbrace{c}_{\text{P.S.}}$$

So hoping if we set $u = \frac{1}{r}$, our diff. eq.

$$-r^2\ddot{r} + \frac{r_0^2 v_0^2}{r} = GM$$

will transform into

$$\frac{d^2u}{d\theta^2} + u = c \quad \ldots$$

$$-r^2\ddot{r} + \frac{r_0^2 v_0^2}{r} = GM$$

Let $u = \frac{1}{r}$

$$\frac{du}{dt} = -\frac{1}{r^2}\cdot\frac{dr}{dt}$$

$$\frac{du}{d\theta}\cdot\frac{d\theta}{dt} =$$

$$\therefore \frac{du}{d\theta}\cdot r^2\dot{\theta} = -\dot{r}$$

$$\therefore \frac{du}{d\theta}\cdot r_0 v_0 = -\dot{r}$$

$$\frac{d}{dt}\left(\frac{d^2u}{d\theta^2}\cdot\frac{d\theta}{dt}\cdot r_0 v_0 = -\ddot{r}\right.$$

$$\therefore \frac{d^2u}{d\theta^2}\cdot r_0^2 v_0^2 = -r^2\ddot{r}$$

$$\Longrightarrow \frac{d^2u}{d\theta^2}\cdot r_0^2 v_0^2 + \frac{r_0^2 v_0^2}{r} = GM$$

$$\therefore \frac{d^2u}{d\theta^2} + u = \frac{GM}{r_0^2 v_0^2}$$

That's what we hoped!!

$$\therefore u = a\cos\theta + b\sin\theta + \frac{GM}{r_0^2 v_0^2}$$

$$\frac{du}{d\theta} = -a\sin\theta + b\cos\theta$$

When $\theta = 0$, $u = \frac{1}{r_0}$, $\frac{du}{d\theta} = 0$.

$$\therefore b = 0, \quad a + \frac{GM}{r_0^2 v_0^2} = \frac{1}{r_0}$$

$$\therefore a = \frac{GM}{r_0^2 v_0^2}\left[\frac{r_0 v_0^2}{GM} - 1\right]$$

$$\therefore \frac{1}{r} = \frac{GM}{r_0^2 v_0^2}\left[\left(\frac{r_0 v_0^2}{GM} - 1\right)\cos\theta + 1\right]$$

Let $e = \frac{r_0 v_0^2}{GM} - 1$, so $r_0(e+1) = \frac{r_0^2 v_0^2}{GM}$

$$r = \frac{r_0^2 v_0^2}{GM} \cdot \frac{1}{1 + e\cos\theta}$$

$$\therefore r = \frac{r_0(1+e)}{1+e\cos\theta} \quad \text{where } e = \frac{r_0 v_0^2}{GM} - 1 = \left(\frac{v_0}{v_{crit}}\right)^2 - 1$$

$$v_{crit} = \sqrt{\frac{GM}{r_0}}$$

General equation for conic section eccentricity e

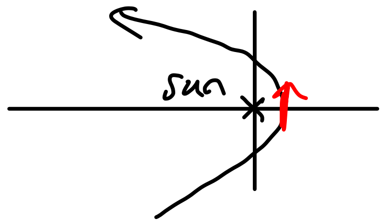
This is our final solution:

$$\boxed{v_{crit} = \sqrt{\frac{GM}{r_0}}, \quad e = \left(\frac{v_0}{v_{crit}}\right)^2 - 1 \quad \text{Then} \quad r = \frac{r_0(1+e)}{1+e\cos\theta}}$$

$$v_{crit} = \sqrt{\frac{GM}{r_0}}, \quad e = \left(\frac{v_0}{v_{crit}}\right)^2 - 1 \quad \text{Then} \quad r = \frac{r_0(1+e)}{1+e\cos\theta}$$

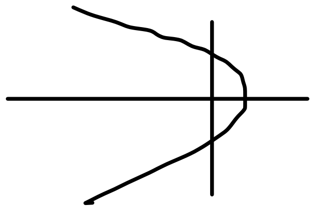
Think about e ...

$v_0 > \sqrt{2}\, v_{crit} \iff e > 1$



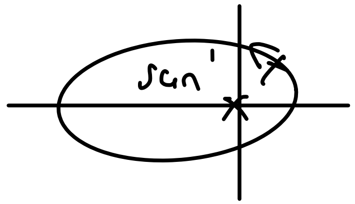
hyperbola, planet escapes sun's gravity to infinity.

$v_0 = \sqrt{2}\, v_{crit} \iff e = 1$



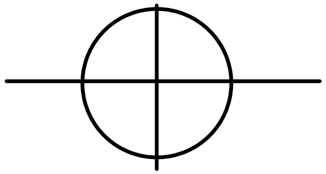
parabola, still escapes

$v_{crit} < v_0 < \sqrt{2}\, v_{crit} \iff 0 < e < 1$



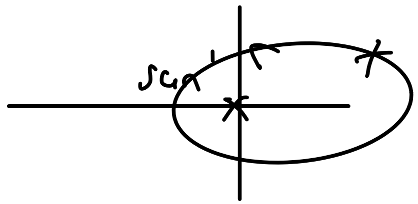
Ellipse ✓

$v_0 = v_{crit} \iff e = 0$



Circle

$0 < v_0 < v_{crit} \iff -1 < e < 0$



Another ellipse, other focus!

We've proved K I from N L of G.

Finally derive KIII ... period T for one full orbit.

Assume $0 \le e < 1$

From KII, know area swept out in time t is $\frac{1}{2} r_0 v_0 t$

Period T is time when area is area of ellipse

$$A = \frac{1}{2}\int_0^{\theta} r^2 \, d\theta = \frac{1}{2}\int_0^{t} \underbrace{r^2 \dot{\theta}}_{r_0 v_0} \, dt = \frac{1}{2} r_0 v_0 t$$

So

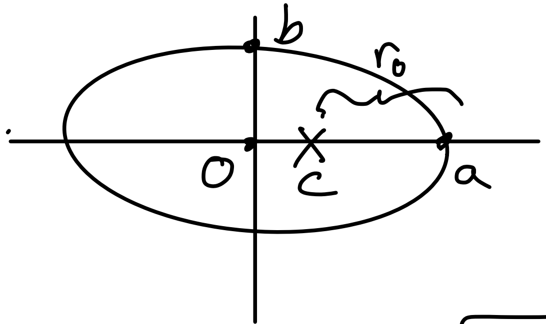
$$\frac{1}{2} r_0 v_0 T = \pi a^2 \sqrt{1-e^2}$$

$$v_0 = v_{circ}\sqrt{1+e} = \frac{\sqrt{GM}\sqrt{1+e}}{\sqrt{r_0}}$$

$$\therefore \frac{\sqrt{r_0}\sqrt{GM}\cancel{\sqrt{1+e}}}{2\cancel{\sqrt{r_0}}} T = \pi a^2 \cancel{\sqrt{1+e}}\sqrt{1-e}$$

$$\therefore \frac{\sqrt{a}\cancel{\sqrt{1-e}}\sqrt{GM}}{2} T = \pi a^2 \cancel{\sqrt{1-e}}$$

$$\boxed{T = \frac{2\pi a^{3/2}}{\sqrt{GM}}}$$



Area $= \pi a b$

$c = ae$

$e = \sqrt{1 - \frac{b^2}{a^2}}$

$\underbrace{b = a\sqrt{1-e^2}}$

$r_0 = a - c$

$= a - ae$

$\therefore \underbrace{r_0 = a(1-e)}$

So : $\boxed{T = \dfrac{2\pi a^{3/2}}{\sqrt{GM}}}$

You can work out mass of sun!!!

KIII $\dfrac{a^3}{T^2} = \underbrace{\dfrac{GM}{4\pi^2}}$

Constant independent of r_0, v_0 — the planet!

Shows $\dfrac{a^3}{T^2}$ is constant for all planets.