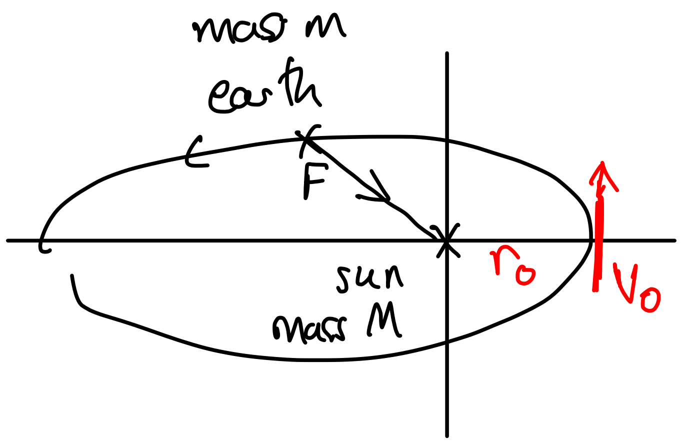
Planetary motion

① K II (equal areas in equal time) $\iff$ $r^2\dot{\theta}$ conserved

$\iff$ central force



② Central force is $\frac{GMm}{r^2}$, some constant G

$\iff$ $r = \frac{r_0(1+e)}{1+e\cos\theta}$ where $e = \left(\frac{V_0}{V_{crit}}\right)^2 - 1$

$\iff$ K I (orbits are ellipses) $V_{crit} = \sqrt{\frac{GM}{r_0}}$

$V_0 = \sqrt{2}\, V_{crit}$ escape velocity

(see HW for more about)

③ $T = \frac{2\pi\, a^{3/2}}{\sqrt{GM}}$, a: major radius of ellipse $\implies$ K III

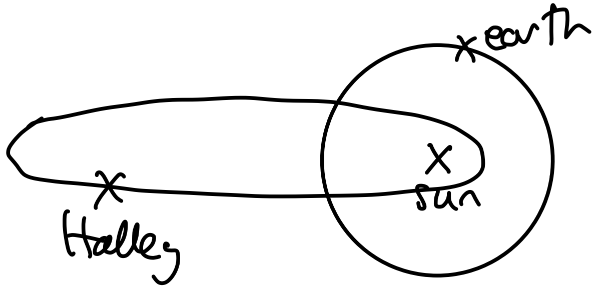
period of orbit

An application

For planets, e is rather close to zero. For earth, $e_{earth} = 0.016$.

So orbit is close to circular.

But Halley's comet has $e_{Halley} = 0.97$



The major radius $a_{Halley} \approx 36\, a_{earth}$

$$T_{Halley} = \frac{2\pi \left(a_{Halley}\right)^{3/2}}{\sqrt{GM}} = \frac{2\pi \left(36\, a_{earth}\right)^{3/2}}{\sqrt{GM}} = 36^{3/2} \underbrace{\frac{2\pi \left(a_{earth}\right)^{3/2}}{\sqrt{GM}}}_{T_{earth} = 1\text{ year}} \approx 76 \text{ yrs}$$

1531, 1607, 1682

Area of ellipse $= \pi a b = \pi a^2 \sqrt{1-e^2} = \dfrac{\pi r_0^2 (1-e)^{1/2}(1+e)^{1/2}}{(1-e)^2} = \boxed{\dfrac{\pi r_0^2 (1+e)^{1/2}}{(1-e)^{3/2}}}$

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $\quad r = \dfrac{r_0(1+e)}{1+e\cos\theta}$ $\quad (0<e<1)$

$e = \sqrt{1-\dfrac{b^2}{a^2}} \implies b = a\sqrt{1-e^2}$

$r_0 = a(1-e)$

Can we prove this formula directly? $\quad \dfrac{1}{2}\displaystyle\int_{\theta_1}^{\theta_2} r^2 \, d\theta$

Its $\quad r_0^2(1+e)^2 \displaystyle\int_0^{\pi} \frac{d\theta}{(1+e\cos\theta)^2}$

We'd better have $\quad \displaystyle\int_0^{\pi} \frac{d\theta}{(1+e\cos\theta)^2} = \frac{\pi}{(1-e^2)^{3/2}}$

How to show this??? $\quad$ Nasty definite integral ...

There's a famous substitution $t = \tan\frac{\theta}{2}$

$$2\arctan t = \theta$$

$$\frac{2}{1+t^2}dt = d\theta$$

$$\left\{\begin{aligned} \cos\theta &= \frac{1-t^2}{1+t^2} \\ \sin\theta &= \frac{2t}{1+t^2} \\ \tan\theta &= \frac{2t}{1-t^2} \end{aligned}\right.$$

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = \frac{2}{\sec^2\frac{\theta}{2}} - 1$$

$$= \frac{2}{1+\tan^2\frac{\theta}{2}} - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$$

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \frac{2t}{1-t^2}$$

This substitution is great for integrals involving $\sin\theta, \cos\theta, \tan\theta \rightsquigarrow$ rational function of t.

eg $\int \sec\theta \cdot d\theta = \int \frac{1}{\cos\theta}\cdot d\theta = \int \frac{\cos\theta}{\cos^2\theta}\cdot d\theta = \int \frac{\cos\theta}{1-\sin^2\theta}\cdot d\theta = \cdots$

$$\int \frac{1}{\cos\theta}\cdot d\theta = \int \frac{\cancel{1+t^2}}{1-t^2}\cdot\frac{2}{\cancel{1+t^2}}\cdot dt = \int \frac{2}{(1+t)(1-t)}dt = \int\left[\frac{1}{1+t} + \frac{1}{1-t}\right]dt$$

$$= \ln(1+t) - \ln(1-t) = \ln\left(\frac{1+t}{1-t}\right) = \ln\left(\frac{(1+t)^2}{1-t^2}\right)$$

$$= \ln\left(\frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2}\right) = \ln(\sec\theta + \tan\theta)$$

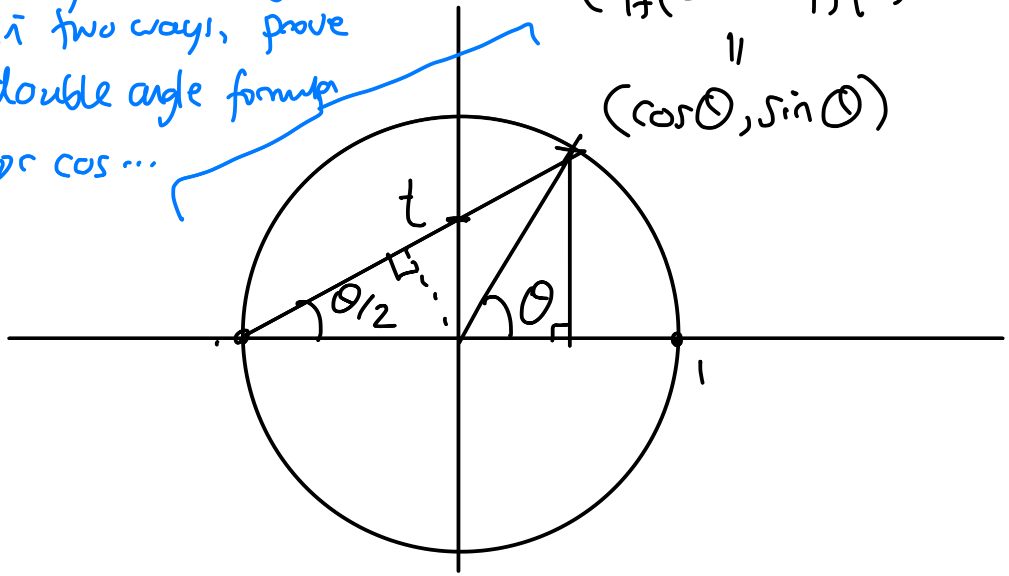
Where does this substitution come from? Geometry

By calculating length in two ways, prove double angle formula for cos...

$$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

$$\parallel$$

$$(\cos\theta, \sin\theta)$$

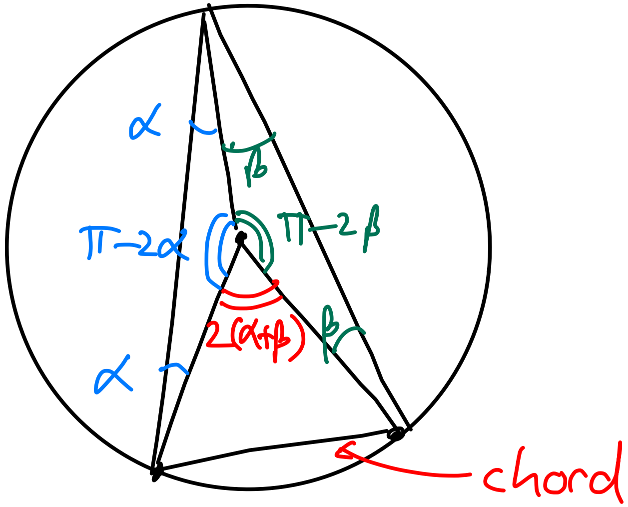


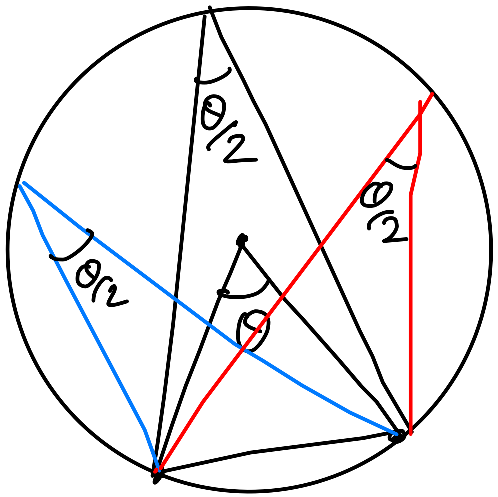
$$t = \tan\frac{\theta}{2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2} \qquad \sin\theta = \frac{2t}{1+t^2}$$

Number t parametrises the unit circle except for point $(-1, 0)$

Geometry of circles





$$\int_0^{\pi} \frac{d\theta}{(1+e\cos\theta)^2}$$

Let $t = \tan\frac{\theta}{2}$, $d\theta = \frac{2}{1+t^2}\,dt$, $\cos\theta = \frac{1-t^2}{1+t^2}$.

$$\int_0^{\infty} \frac{1}{\left[1+e\,\frac{1-t^2}{1+t^2}\right]^2} \frac{2(1+t^2)}{(1+t^2)^2}\,dt = 2\int_0^{\infty} \frac{1+t^2}{\left[1+t^2+e(1-t^2)\right]^2}\,dt$$

$$= 2\int_0^{\infty} \frac{1+t^2}{\left[(1+e)+(1-e)t^2\right]^2}\,dt = \frac{2}{(1+e)^2}\int_0^{\infty} \frac{1+t^2}{\left[1+\frac{1-e}{1+e}t^2\right]^2}\,dt$$

Let $\frac{1-e}{1+e}t^2 = \tan^2\varphi$, $\sqrt{\frac{1-e}{1+e}}\,t = \tan\varphi$, $dt = \sqrt{\frac{1+e}{1-e}}\sec^2\varphi\,d\varphi$

$$= \frac{2}{(1+e)^2}\cdot\sqrt{\frac{1+e}{1-e}}\int_0^{\pi/2} \frac{1+\frac{1+e}{1-e}\tan^2\varphi}{(\cancel{1+\tan^2\varphi})^2}_{\sec^2\varphi}\sec^2\varphi\,d\varphi = \frac{2}{(1+e)^{3/2}(1-e)^{3/2}}\int_0^{\pi/2}\left[(1-e)+(1+e)\tan^2\varphi\right]\cos^2\varphi\,d\varphi$$

$$= \frac{2}{(1-e^2)^{3/2}} \int_0^{\pi/2} \left[(1-e)\cos^2\varphi + (1+e)\sin^2\varphi\right] d\varphi$$

$$= \frac{2}{(1-e^2)^{3/2}} \int_0^{\pi/2} \left[1 - e\,\overbrace{(\cos^2\varphi - \sin^2\varphi)}^{\cos 2\varphi}\right] d\varphi$$

$$= \frac{2}{(1-e^2)^{3/2}} \left[\varphi - \frac{e}{2}\sin 2\varphi\right]_0^{\pi/2}$$

$$= \frac{\pi}{(1-e^2)^{3/2}}$$