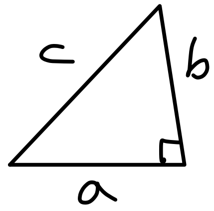
Digression — Pythagorean triples ← (a,b,c) positive integers with $a^2+b^2=c^2$



eg $(3,4,5)$ $(5,12,13)$

How to find them all?

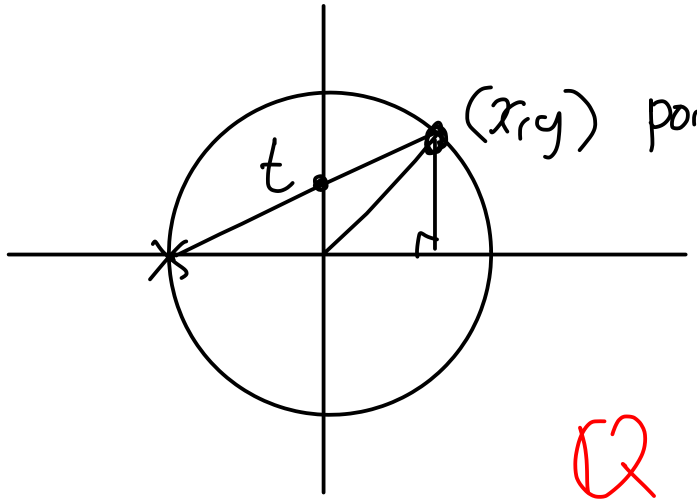
Divide by c^2 ... $\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2=1$ } $x=\frac{a}{c}\in\mathbb{Q}$ $y=\frac{b}{c}\in\mathbb{Q}$

$$x^2+y^2=1$$

Equivalent problem is to find rational pairs $(x,y)\in\mathbb{Q}^2$ on unit circle.

eg $\left(\frac{3}{5},\frac{4}{5}\right)$ $\left(\frac{5}{13},\frac{12}{13}\right)$

We know how to do this using the trick we saw last time came from substitution $t=\tan\frac{\theta}{2}$,



(x,y) point on unit circle

$x^2+y^2=1$

$$x=\frac{1-t^2}{1+t^2},\quad y=\frac{2t}{1+t^2}$$

$$\Updownarrow$$

$$t=\frac{y}{1+x}$$

We have functions

$$\overset{\mathbb{Q}}{\cancel{\mathbb{R}}} \rightleftarrows \left\{(x,y)\in \overset{\mathbb{Q}^2}{\cancel{\mathbb{R}^2}} \;\middle|\; x^2+y^2=1,\ x\neq -1\right\}$$

$$t \longmapsto \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

$$\frac{y}{1+x} \longleftarrow\!\!\mid (x,y)$$

These are mutual inverses ... so they are bijections.

Thus, $\mathbb{Q}$ parametrises rational points on unit circle, hence, Pythagorean triples.

eg $t=\frac{1}{2} \rightsquigarrow \left(\frac{3}{5}, \frac{4}{5}\right)$ $\quad t=\frac{2}{3} \rightsquigarrow \left(\frac{5}{13}, \frac{12}{13}\right)$ $\quad t=\frac{3}{4} \rightsquigarrow \left(\frac{7}{25}, \frac{24}{25}\right)$

Main topic this week How to define classical functions properly.

ln, exp — trig — hyperbolic function

We've talked about these before.

- $y = \ln x \iff x = e^y$ inverse functions
- laws of logs / exponents
- $a^x = (e^{\ln a})^x = e^{x \ln a}$ (definition!)

  $(a > 0, x \in \mathbb{R})$

First approach Start with $\ln x$ ...

$$\ln x = \int_1^x \frac{1}{t}\, dt$$

definition!

— makes sense as $\frac{1}{t}$ is continuous

— $\frac{d}{dx}(\ln x) = \frac{1}{x}$

} FTC

As $\frac{1}{x} > 0$ for $x > 0$, $\ln x$ is increasing, hence, 1–1, so invertible. Then you define e^x to be the inverse function.

Prove some properties of $\ln x / e^x$ from these definitions.

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad \frac{d}{dx}(e^x) = e^x$$

Let $y = e^x$, so $\ln y = x$

$$\therefore \frac{d}{dx}(\ln y) = 1$$

$$\therefore \frac{1}{y} \cdot \frac{dy}{dx} =$$

$$\therefore \frac{dy}{dx} = y = e^x$$

Laws of log

- $\ln(ab) = \ln a + \ln b$
- $\ln(a^b) = b \ln a$

Proof: By the definition, we need to show

$$\int_1^{ab} \frac{1}{t}\, dt = \int_1^a \frac{1}{t}\, dt + \int_1^b \frac{1}{t}\, dt.$$

Take RHS ... $\int_1^a \frac{1}{t}\, dt + \underbrace{\int_1^b \frac{1}{t}\, dt}_{\text{Let } u = at} = \int_1^a \frac{1}{t}\, dt + \int_a^{ab} \frac{\cancel{a}}{u} \frac{1}{\cancel{a}}\, du$

$$= \int_1^a \frac{1}{t}\, dt + \int_a^{ab} \frac{1}{t}\, dt = \int_1^{ab} \frac{1}{t}\, dt = LHS$$

- $\ln(a^b) = \ln(e^{b \ln a})$
  $= b \ln a$

Laws of exponents

- $e^{a+b} = e^a e^b$
- $(e^a)^b = e^{ab}$

Proof
- $RHS = e^a e^b = e^{\ln(e^a e^b)} = e^{\ln(e^a)+\ln(e^b)} = e^{a+b} = LHS$ ✓
- $LHS = (e^a)^b = e^{b\ln(e^a)} = e^{ab} = RHS$ ✓

All good! ... $\ln x = \int_1^x \frac{1}{t}\,dt$ very clean and tidy

e^x inverse function a bit indirect. ← e^x is easier than $\ln x$

How can we define e^x more directly?

Second approach

Could try to define e^x to be the unique solution to the differential equation

$$f'(x) = f(x) \quad \text{with} \quad f(0) = 1$$

This is good ... prove properties like $e^{x+y} = e^x e^y$ quite easily.

Indeed, let $f(x) = \dfrac{e^{x+y}}{e^y}$ (y constant)

$$f'(x) = \frac{e^{x+y}}{e^y} = f(x), \quad f(0) = \frac{e^y}{e^y} = 1$$

$\therefore f(x) = e^x$

$\therefore \dfrac{e^{x+y}}{e^y} = e^x$

$\therefore e^{x+y} = e^x e^y$ ✓

Think about this as a recursive recipe for computing a power series

Why does any solution to this diff. eq. exist???

!EXISTENCE!

<u>Third approach</u> Use power series.

This is a major theory which needs to be developed carefully at this point.

(eg) Geometric series $1+x+x^2+x^3+\dots = \frac{1}{1-x}$ providing $|x|<1$.

In general, a power series is

$$f(x) = a_0+a_1x+a_2x^2+a_3x^3+\dots = \sum_{n=0}^{\infty} a_n x^n$$ for sequence $a_0, a_1, a_2, a_3, \dots$ of real no.s

means: $\lim_{N\to\infty} \sum_{n=0}^{N} a_n x^n$ ⟵ may not converge.

The $x \in \mathbb{R}$ for which it does converge give the <u>domain</u> of the function $f(x)$ defined by the power series.

<u>Theorem</u> Given a power series $f(x)$ as above, there's $R \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ called the <u>radius of convergence</u> of $f(x)$ so that $f(x)$ converges absolutely for all x with $|x|<R$ and doesn't converge for any x with $|x|>R$. On $(-R,R)$, $f(x)$ is differentiable with $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$, with series also converging on $(-R,R)$.

Note if $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ has $R > 0$

then $a_0 = f(0)$, $a_1 = f'(0)$, $a_2 = \frac{f''(0)}{2!}$, ..., $a_n = \frac{f^{(n)}(0)}{n!}$, ...

for example, $f(x) = e^x$ ... if we could define it by a power series,

this would show all $a_n = \frac{1}{n!}$

$\cancel{e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots}$

$\int_1^e \frac{1}{t} \, dt = 1$

Definition $\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}$

Note this power series has $R = \infty$ (it converges $\forall x$ ... ratio test)

hence it gives an ∞-differentiable function with domain $\mathbb{R}$.

$\frac{d}{dx}(e^x) = e^x$, $e^0 = a_0 = 1$ ... gives a function

solving the diff. eq. from approach two.