Power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots = \sum_{n=0}^{\infty} a_n x^n$$

Radius of convergence $R \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ $\longrightarrow$ Series converges absolutely for $|x| < R$
Series diverges for $|x| > R$.

On $(-R, R)$, $f(x)$ is differentiable with $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$, which converges on $(-R, R)$

$$\boxed{a_n = f^{(n)}(0)/n!}$$

<u>Idea</u>: a differential equation is a recursive recipe for computing a power series.

$$f'(x) = f(x), \; f(0) = 0 \implies a_n = \frac{1}{n!} \implies f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

e^x

Then this power series becomes the definition of exponential function.

Ratio test $\implies R = \infty$ so e^x has domain $\mathbb{R}$

Given a function which does satisfy the diff. eq.

$$e^x e^y = e^{x+y}$$

→ One way was by $\ln(ab) = \ln(a) + \ln(b)$

→ Another was using diff. eq.

Cauchy product

Depends on absolute convergence

Third way

$$e^x e^y = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots\right)$$

foil

$$= 1 + (x+y) + \left(\frac{x^2}{2!} + xy + \frac{y^2}{2!}\right) + \left(\frac{x^3}{3!} + \frac{x^2}{2!} \cdot y + x \cdot \frac{y^2}{2!} + \frac{y^3}{3!}\right) + \cdots$$

$$= 1 + (x+y) + \frac{1}{2!}\left(x^2 + 2xy + y^2\right) + \frac{1}{3!}\left(x^3 + 3x^2y + 3xy^2 + y^3\right) + \cdots$$

$$= 1 + (x+y) + \frac{(x+y)^2}{2!} + \frac{(x+y)^3}{3!} + \cdots$$

$$= e^{x+y}$$

What about ~~ln~~ x ? Could we define that by power series?

$$\boxed{\ln x := \int_1^x \frac{1}{t}\,dt}$$

domain $(0,\infty)$

$$f(x) = \ln(1+x)$$

$$f'(x) = \frac{1}{1+x}, \quad f(0) = 0$$

If $f(x) = \cancel{a_0} + a_1 x + a_2 x^2 + \cdots$ then $f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

$$= 1 - x + x^2 - x^3 + \cdots \quad \text{on } |x| < 1$$

$a_0 = f(0) = 0$

$$\therefore a_1 = 1, \quad a_2 = -\frac{1}{2}, \quad a_3 = \frac{1}{3}, \ldots$$

$$\therefore f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Problem: This series has $R = 1$ so only gives a function on domain $(-1, 1)$

$\Rightarrow$ only get $\ln(x)$ on domain $(0, 2)$ ✗ Not a good approach.

Example My favorite. Due to Newton ~ 1665

Binomial series. $\leftarrow$ Binomial theorem (classical) $\alpha \in \mathbb{N} = \{0,1,2,\dots\}$

$$(1+x)^\alpha = \sum_{n=0}^{\alpha} \binom{\alpha}{n} x^\alpha$$

Binomial coefficient from Pascal's Δ

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ \vdots \end{array}$$

Definition Generalized binomial coefficient

$\alpha \in \mathbb{R}$, $n \in \mathbb{N}$

$$\binom{\alpha}{n} = \frac{\overbrace{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}^{n \text{ terms}}}{n!}, \qquad \binom{\alpha}{0} = 1.$$

Lemma $\binom{\alpha-1}{n} + \binom{\alpha-1}{n-1} = \binom{\alpha}{n}$ $\leftarrow$ When $\alpha \in \mathbb{N}$ is Pascal Δ recursion.

Proof LHS $= \dfrac{(\alpha-1)(\alpha-2)\cdots(\alpha-n)}{n!} + \dfrac{(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)\,n}{(n-1)!\,n}$

$$= \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)[(\alpha-n)+n]}{n!} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} = \binom{\alpha}{n} = \text{RHS}$$

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$\Longrightarrow$ When $\alpha \in \mathbb{N}$, generalized binomial cf is the usual one from Pascal's Δ.

Consider binomial series

$$f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

If $\alpha \in \mathbb{N}$, $\binom{\alpha}{n} = 0$ for $n > \alpha$

The infinite sum collapses to

$$\sum_{n=0}^{\alpha} \binom{\alpha}{n} x^n = (1+x)^\alpha$$

Show R, radius of convergence, is ≥ 1.

Use ratio test to show series converges if $|x| < 1$

$$\left| \frac{\binom{\alpha}{n+1} x^{n+1}}{\binom{\alpha}{n} x^n} \right| = \left| \frac{\alpha(\alpha-1)\cdots(\alpha-n)\, x^{n+1}\, n!}{(n+1)!\, \alpha(\alpha-1)\cdots(\alpha-n+1)\, x^n} \right| = \left| \frac{(\alpha-n)x}{n+1} \right|$$

$$= \left| \frac{\frac{\alpha}{n} - 1}{1 + \frac{1}{n}} \right| |x| \longrightarrow |x| < 1 \quad \text{as } n \to \infty$$

So this series gives a well-defined, differentiable function on $(-1, 1)$.

$$f'(x) = \sum_{n=1}^{\infty} n \binom{\alpha}{n} x^{n-1} = \sum_{n=1}^{\infty} \frac{\cancel{n}\,\alpha(\alpha-1)\cdots(\alpha-n+1)}{\cancel{n!}\;(n-1)!} x^{n-1} \quad \leftarrow \text{let } n' = n-1$$

$$\therefore f'(x) = \alpha \sum_{n'=0}^{\infty} \binom{\alpha-1}{n'} x^{n'} = \alpha \sum_{n=0}^{\infty} \binom{\alpha-1}{n} x^{n}$$

$$\therefore (1+x) f'(x) = \alpha \sum_{n=0}^{\infty} \binom{\alpha-1}{n} x^{n} + \alpha \sum_{n=0}^{\infty} \binom{\alpha-1}{n} x^{n+1}$$

$$= \alpha \sum_{n=0}^{\infty} \binom{\alpha-1}{n} x^{n} + \alpha \sum_{n=1}^{\infty} \binom{\alpha-1}{n-1} x^{n} \quad \Big\} \text{ limit laws}$$

$$= \alpha \sum_{n=0}^{\infty} \left[\binom{\alpha-1}{n} + \binom{\alpha-1}{n-1} \right] x^{n} \quad \Big\} \text{ lemma}$$

$$= \alpha \sum_{n=0}^{\infty} \binom{\alpha}{n} x^{n} = \alpha f(x)$$

Shows: $(1+x) f'(x) = \alpha f(x)$

So: $f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ on $(-1, 1)$ satisfies diff. eq.

$$(1+x) f'(x) = \alpha f(x), \qquad f(0) = 1$$

Now solve this by separation of variable to work out what function $f(x)$ is ...

$$(1+x) \frac{dy}{dx} = \alpha y$$

$$\therefore \int \frac{dy}{y} = \int \frac{\alpha}{1+x} \, dx + c$$

$$\therefore \ln(y) = \alpha \ln(1+x) + c$$

When $x=0$, $y=1$ so $c=0$

$$\therefore \ln(y) = \ln\left[(1+x)^{\alpha}\right]$$

$$\therefore y = (1+x)^{\alpha}$$

$\boxed{\text{Theorem For any } \alpha \in \mathbb{R}, \ (1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \text{ on } (-1, 1)}$

eg $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$ recovers geometric series

$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots$

$$\binom{1/2}{3} = \frac{1/2\,(1/2-1)(1/2-2)}{3!} = \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)}{3\cdot 2} = \frac{1}{16}$$

<u>Taylor's theorem</u> If $f(x)$ is a function which is ∞-ly differentiable at $x=0$ and $a_n = \frac{f^{(n)}(0)}{n!}$ then

$$f(x) = \underbrace{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}_{n\text{-th Taylor polynomial}} + E(x)$$

where $E(x) = \frac{f^{(n+1)}(t)}{(n+1)!} x^{n+1}$ for some t with $|t| < |x|$.