Formal definition of trig. functions

So far our approach to trig function was naive — from SOHCAHTOA

We had a theorem (lecture 3-3):-

Theorem If $f''(x) + f(x) = 0$ then $f(x) = f(0)\cos x + f'(0)\sin x$

This is really a uniqueness theorem. We didn't ever worry about existence of solutions — what are $\cos x, \sin x$ in first place???

Expect: $\cos x$ is unique solution to this diff. eq. with $f(0)=1,\ f'(0)=0$

$\sin x$ " " " " " " " $f(0)=0,\ f'(0)=1$

Now we're going to start from scratch.

<u>Def</u> $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Both series have $R = \infty$, so domains of these functions are all of $\mathbb{R}$.

General theory of power series $\Rightarrow$ $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$

These functions therefore do give solutions to $f''(x) + f(x) = 0$ and they satisfy the expected initial conditions

<u>Digression</u> You can use Taylor's theorem to give a new proof of that uniqueness theorem

<u>Sketch</u> Suppose $f''(x) + f(x) = 0$. Then $f(x)$ is ∞-differentiable, so we can apply Taylor's theorem to it. We deduce that

$f(x) =$ (nth degree Taylor poly) $+ \; E_n(x)$ (Taylor remainder)

$$= f(0)\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right] + f'(0)\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right] + E_n(x)$$

stop when you hit x^n

where $|E_n(x)| = \left| \frac{f^{(n+1)}(t)}{(n+1)!} x^{n+1} \right|$ some t with $0 < |t| < |x|$

As $f(x)$ is differentiable, it's cts, so bounded on $[-|x|, |x|]$ as is $f'(x)$. Pick N so $|f(t)|, |f'(t)| \leq N$ for such t

Then $|E_n(x)| \leq \frac{N}{(n+1)!} |x|^{n+1} \longrightarrow 0$ as $n \to \infty$.

Shows that $f(x) = f(0)\cos x + f'(0)\sin x$, shows the uniqueness //

We've defined $\cos x$, $\sin x$ by power series.

① $\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

This follows from the diff. eq. (HW 3)

<u>Sketch</u> for sin, let $f(x) = \sin(x+y)$ some constant y

So $f''(x) + f(x) = 0$

$\therefore f(x) = f(0)\cos x + f'(0)\sin x = \sin y \cos x + \cos y \sin x$ //

② Note $\cos x$ is even, $\sin x$ is odd

$\cos(-x) = \cos x$ $\qquad$ $\sin(-x) = -\sin x$

$\cos(x-x) = \cos^2 x + \sin^2 x$

$\therefore \quad 1 = \cos^2 x + \sin^2 x \implies$ "Pythagoras"

$\implies |\cos x| \leq 1, \quad |\sin x| \leq 1$

(3) Let's estimate $\cos 2$ using degree 3 Taylor polynomial.

$$\cos 2 = 1 - \frac{2^2}{2!} + E \qquad |E| = \frac{|\cos t|\, 2^4}{4!} \text{ some } 0<t<2$$

$$\leq 1 - 2 + \frac{2}{3} = -\frac{1}{3} \qquad \leq \frac{2^4}{4!} = \frac{16}{24} = \frac{2}{3}$$

Shows $\cos 2 < 0$, $\cos 0 > 1$

As $\cos x$ is cts, Intermediate Value Theorem $\Rightarrow$ $\exists$ some $x \in (0,2)$ with $\cos x = 0$.

<u>Def</u> Let π be smallest positive real number such that $\cos\left(\frac{\pi}{2}\right) = 0$ (In fact $0 < \pi < 4$).

(4) As $\cos x > 0$ for $0 \leq x < \frac{\pi}{2}$, and $\cos x = \sin' x$, shows $\sin x$ is increasing on $\left[0, \frac{\pi}{2}\right]$. As $\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$

$\cos \pi = \cos^2 \frac{\pi}{2} - \sin^2 \frac{\pi}{2} = -1$

$\sin \pi = 2 \sin \frac{\pi}{2} \cos \frac{\pi}{2} = 0$

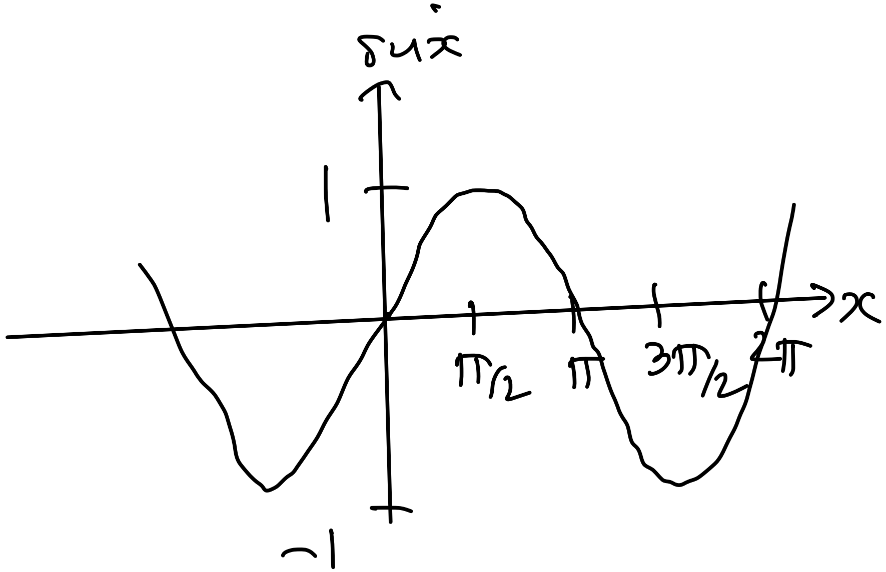
$\cos 2\pi = \cos^2 \pi - \sin^2 \pi = 1$

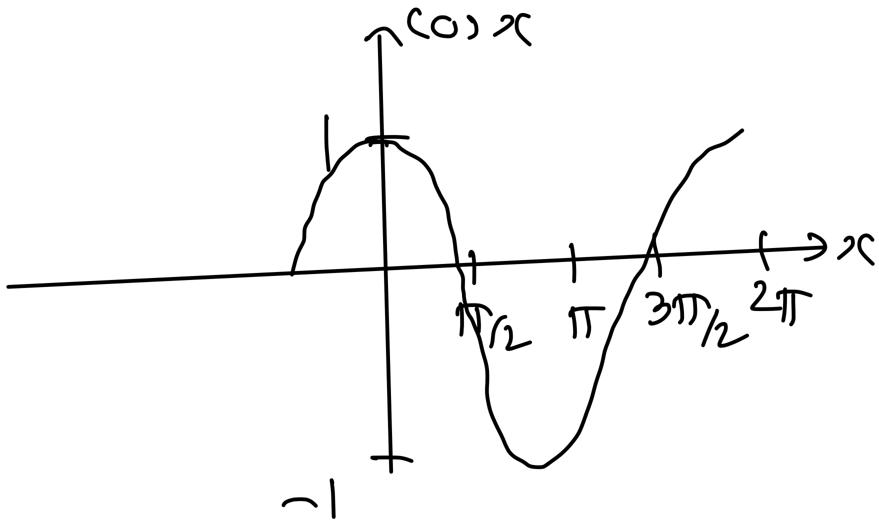
$\sin 2\pi = 2 \sin \pi \cos \pi = 0$

$\cos(x + 2\pi) = \cos x$

$\sin(x + 2\pi) = \sin x$

$\Rightarrow$ $\cos x$, $\sin x$ are periodic, period 2π





roughly graphs we were expecting !!

(5) sin x is increasing on $[0, \pi/2]$, and odd, so increasing on $[-\pi/2, \pi/2]$. Increasing $\Rightarrow$ 1-1 $\Rightarrow$ invertible.

<u>Def</u> arcsin to be the inverse function of sin on $[-\pi/2, \pi/2]$.

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad y = \arcsin x \qquad \therefore \sin y = x$$

$$\therefore \cos y \cdot \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \cancel{\pm}\, + \frac{1}{\sqrt{1-\sin^2 y}}$$

(6) $$\int \sqrt{1-x^2}\, dx = \int \sqrt{1-\sin^2\theta}\, \cos\theta \, d\theta$$

$x = \sin\theta$

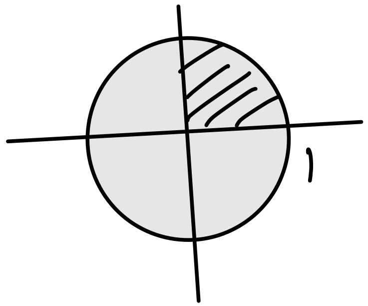
$dx = \cos\theta\, d\theta$

$$= \int \cos^2\theta\, d\theta = \frac{1}{2}\int (\cos 2\theta + 1)\, d\theta$$

$$= \frac{1}{4}\sin 2\theta + \frac{\theta}{2} = \frac{\sin\theta\cos\theta}{2} + \frac{\theta}{2}$$

$$= \frac{x\sqrt{1-x^2}}{2} + \frac{\arcsin x}{2}$$

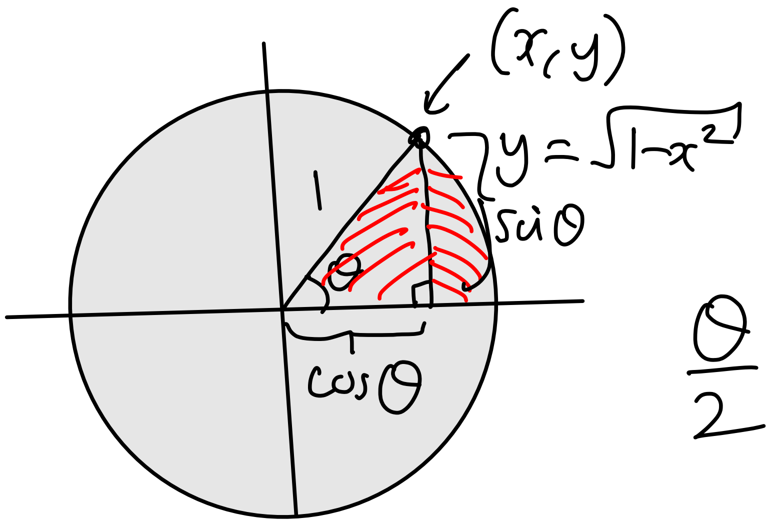
⑦ Show the area of unit circle is π



$$= 4\int_0^1 \sqrt{1-x^2}\,dx = \left[2x\sqrt{1-x^2} + 2\arcsin x\right]_0^1$$

$$= 2\arcsin 1 = \pi \quad ✓$$

⑧ Prove SOHCAHTOA



Right angled △ angle θ.

Angle θ means sector is $\frac{\theta}{2\pi}$ of entire ⊙,

so sector area $= \frac{\theta}{2\pi} \times \pi = \frac{\theta}{2}$

$$\frac{\theta}{2} = \frac{1}{2}x\sqrt{1-x^2} + \int_x^1 \sqrt{1-t^2}\,dt$$

$$= \frac{1}{2}x\sqrt{1-x^2} + \left[\frac{1}{2}t\sqrt{1-t^2} + \frac{1}{2}\arcsin t\right]_x^1$$

$$\frac{\theta}{2} = \frac{\pi}{4} - \frac{1}{2}\arcsin x$$

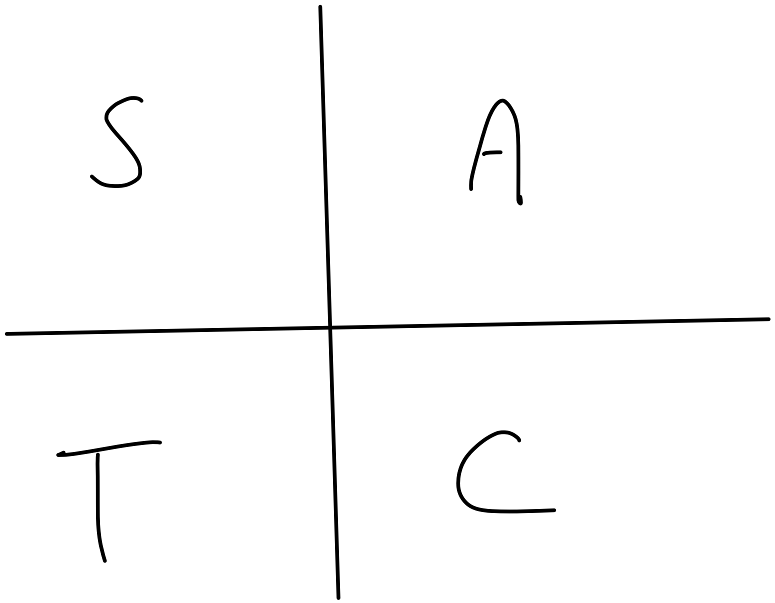
$$\tan\theta := \frac{\sin\theta}{\cos\theta}$$

$$\theta = \frac{\pi}{2} - \arcsin x$$

$$\therefore \arcsin x = \frac{\pi}{2} - \theta$$

$$\therefore \quad x = \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

✓ What we needed for SOHCAHTOA!!



This completes rigorous definition of trig functions.