Introduction to Lie Theory
Homework #1

1. Let $\mathbb{G}_a$ and $\mathbb{G}_m$ be the *additive group* and the *multiplicative group*, respectively, that is, the groups $(\Bbbk, +)$ and $(\Bbbk^\times, \cdot)$ with the obvious affine variety structures.

   (a) Show that the coodinate algebra $\Bbbk[\mathbb{G}_a]$ is isomorphic to the algebra $\Bbbk[T]$ viewed as a Hopf algebra with comultiplication $T \mapsto T \otimes 1 + 1 \otimes T$, counit $T \mapsto 0$ and antipode $T \mapsto -T$.

   (b) Give a similarly explicit description of $\Bbbk[\mathbb{G}_m]$.

   (c) Show that $\mathbb{G}_a$ is isomorphic to the closed subgroup U of $GL_2(\Bbbk)$ consisting of the upper unitriangular matrices.

   (d) Show that the closed subgroup T of $GL_n(\Bbbk)$ consisting of diagonal invertible matrices is isomorphic to $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$ (n times).

2. Let $X_i\,(i \in I)$ be a family of irreducible affine varieties and $\phi_i : X_i \to G$ be morphisms to an algebraic group G. Assume that $e \in \phi_i(X_i)$ for each i[1]. Let H be the subgroup of G generated by the images of all ϕ_i. Show that H is a closed, connected subgroup of G.

   (*Hints.* Let $Y_i = \phi_i(X_i)$. Choose $i_1, \ldots, i_n \in I$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$ so that the irreducible closed subset $\overline{Y_{i_1}^{\varepsilon_1} \cdots Y_{i_n}^{\varepsilon_n}}$ of G is of maximal dimension amongst all such subsets. Then show that $H = \overline{Y_{i_1}^{\varepsilon_1} \cdots Y_{i_n}^{\varepsilon_n}}$. You will also need to use the fact that an algebraic group is equal to the set-wise product UV of any dense open subsets U and V, and that the image of a morphism contains a non-empty open subset of its closure.)

3. Let V be a symplectic vector space, that is, it is a finite-dimensional vector space equipped with a non-degenerate skew-symmetric bilinear form $(\cdot, \cdot)$. The *symplectic group* is the isometry group:

   $$Sp(V) := \{g \in GL(V) \mid (gv, gw) = (v, w) \text{ for all } v, w \in V\}.$$

   (a) Explain why $Sp(V)$ is a closed subgroup of $GL(V)$. Hence, it is an algebraic group.

[1] This hypothesis is necessary: see Exercise 2.2.9(2) of "Linear Algebraic Groups" by T. A. Springer.

(b) Let $\det : GL(V) \to \mathbb{G}_m$ be the morphism of algebraic groups defined by determinant. Show that $Sp(V) \subseteq SL(V) := \ker \det$. (*Hint:* Pfaffians.)

(c) For $0 \neq a \in V$ and $t \in \mathbb{k}$, let $u_a(t) : V \to V$ be the *transvection* $v \mapsto v + t(v, a)a$. Check that $u_a(t) \in Sp(V)$ and that $u_a(t)u_a(t') = u_a(t + t')$.

(d) Let $\phi_a : \mathbb{G}_a \to Sp(V), t \mapsto u_a(t)$. Show that this is a morphism of algebraic groups. Use question 2 and a fact from group theory, namely, that $Sp(V)$ is generated by all transvections, to prove that $Sp(V)$ is connected.

(For an orthogonal vector space V and $\operatorname{char} \mathbb{k} \neq 2$, the isometry group is the *orthogonal group* $O(V)$; in characteristic 2 the definition of $O(V)$ is slightly more complicated. Unlike for $Sp(V)$, elements of $O(V)$ have can determinant either $+1$ or -1. You get the *special orthogonal group* by taking just the ones of determinant $+1$. In fact, $SO(V)$ is the connected component of the identity in $O(V)$. This can be proved similarly to the above[2].)

4. Let G be a connected algebraic group. Use question 2 to prove that the derived subgroup G', that is, the subgroup generated by all commutators $ghg^{-1}h^{-1}$ is a closed, connected subgroup of G. Hence, all of the subgroups in the derived series of G are closed and connected.

   Give an explicit description of the derived series of the closed subgroup B of $GL_n(\mathbb{k})$ consisting of all upper triangular invertible matrices.

5. A *representation* of an algebraic group G is a finite-dimensional vector space V plus a morphism of algebraic groups $\rho : G \to GL(V)$.

   (a) Suppose that V is a representation of G with basis $v_1, \ldots, v_n$. By considering the comorphism ρ^*, or otherwise, show that the functions $f_{i,j} : G \to \mathbb{k}$ defined from $gv_j = \sum_{i=1}^n f_{i,j}(g)v_i$ belong to $\mathbb{k}[G]$. Moreover, the comultiplication of $\mathbb{k}[G]$ sends $f_{i,j} \mapsto \sum_{k=1}^n f_{i,k} \otimes f_{k,j}$, and its counit sends $f_{i,j} \mapsto \delta_{i,j}$.

   (b) Suppose you are given $f_{i,j} \in \mathbb{k}[G]$ for $1 \leq i, j \leq n$ such that the comultiplication and counit of $\mathbb{k}[G]$ have the properties formulated in

[2]See also Exercise 2.2.2(2) of Springer for another approach in characteristic $\neq 2$.

(a). Show that the function $\rho : G \to GL_n(\Bbbk), g \mapsto (f_{i,j}(g))_{1 \leq i,j \leq n}$ is a morphism of algebraic groups.

(This question hints at the notion of a $\Bbbk[G]$-*comodule* which is the appropriate algebraic gadget that corresponds to representations of G.)