Introduction to Lie Theory
Homework #2

1. Calculate:

   (a) The differential $dm_{(e,e)} : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}$ of the multiplication map $m : G \times G \to G$ for any algebraic group G.

   (b) The differential $di_e : \mathfrak{g} \to \mathfrak{g}$ of the group inversion map.

   (c) The differential $d(\det)_e : \mathfrak{gl}_n(\Bbbk) \to \Bbbk$ of the determinant morphism $\det : GL_n(\Bbbk) \to \mathbb{G}_m$.

2. Assume in this question that $p \neq 2$.

   (a) In class, we identified the Lie algebra of the group $Sp_{2n}(\Bbbk)$ with a subalgebra $\mathfrak{sp}_{2n}(\Bbbk)$ of $\mathfrak{gl}_{2n}(\Bbbk)$, and used this to deduce that $\dim Sp_{2n}(\Bbbk) = 2n^2+n$. Do the same thing with the group $SO_n(\Bbbk)$.

   (b) Now let S be the affine variety of $n \times n$ symmetric matrices. Note that $G := SL_n(\Bbbk)$ acts on S via the map $G \times S \to S, (g, x) \mapsto g^T xg$ (a morphism of varieties). Use this plus the following general fact to calculate the dimension of $SO_n(\Bbbk)$ in a different way: the dimension of an orbit of a connected algebraic group is equal to the dimension of the group minus the dimension of the identity component of the stabilizer of a point in the orbit.

3. Recall the notion of a representation of an algebraic group G from HW1, Q5. A *right* $\Bbbk[G]$-*comodule* is a vector space V together with a *structure map* $\eta : V \to V \otimes \Bbbk[G]$, that is, a linear map such that $(\mathrm{id} \otimes \Delta) \circ \eta = (\eta \otimes \mathrm{id}) \circ \eta$ and $(\mathrm{id} \bar{\otimes} \varepsilon) \circ \eta = \mathrm{id}$, where Δ and ε denote the comultiplication and counit on the coordinate algebra $\Bbbk[G]$, respectively.

   (a) For a finite-dimensional right $\Bbbk[G]$-comodule V, show that there is a well-defined representation $\rho : G \to GL(V)$ such that

$$\rho(g)(v) = \sum_{i=1}^{n} f_i(g) v_i$$

   for $g \in G$ and $v \in V$ with $\eta(v) = \sum_{i=1}^{n} v_i \otimes f_i$.

(b) Let $\mathrm{Rep}(G)$ and $\mathrm{comod_{fd}}\text{-}\Bbbk[G]$ denote the categories of representations of G and of finite-dimensional right $\Bbbk[G]$-comodules, respectively, with the obvious notion of morphisms in each case. Show that these categories are *isomorphic*.

4. Let $G := \mathbb{G}_m$ with coordinate algebra $\Bbbk[G] = \Bbbk[T, T^{-1}]$. Let $\rho : G \to V$ be a representation. For $n \in \mathbb{Z}$ let

$$V_n := \{v \in V \mid \rho(t)(v) = t^n v \text{ for all } t \in G\}.$$

Prove that $V = \bigoplus_{n\in\mathbb{Z}} V_n$. Hence, the Abelian category $\mathrm{Rep}(G)$ is semisimple, i.e., all representations decompose into a direct sum of irreducible representations.

(*Hint.* One way to proceed is to let $\eta : V \to \Bbbk[T, T^{-1}]$ be its comodule structure map. Assuming this sends $v \in V$ to $\sum_{n\in\mathbb{Z}} v_n \otimes T^n$, show that $v = \sum_{n\in\mathbb{Z}} v_n$ and $v_n \in V_n$.)

5. An algebraic group G is said to be *linearly reductive* if the category $\mathrm{Rep}(G)$ is semisimple. For example, by Maschke's theorem, a finite group G (viewed as an algebraic group over our usual field $\Bbbk$ of characteristic p) is linearly reductive if and only if either $p = 0$ or $p \nmid |G|$.

(a) Show that $G \times H$ is linearly reductive if and only if both G and H are linearly reductive. Deduce from this and the previous question that any *torus* $T \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m$ (n times) is linearly reductive.

(b) Show that G is linearly reductive if and only if both G° and G/G° are linearly reductive.

(It is known that a connected algebraic group is linearly reductive if and only if $p > 0$ and G is a torus, or $p = 0$ and G is reductive, i.e., $G = TG'$ for a torus $T \leq Z(G)$ and a semisimple group $G' \leq G$.)