Introduction to Lie Theory
Homework #6

1. Let \( g = g_1 \oplus g_2 \) be a direct sum of two Lie algebras. Let \( V \) be a finite-dimensional \( g \)-module. Prove that \( V \) is completely reducible as a \( g \)-module if and only if it is completely reducible both as a \( g_1 \)-module and as a \( g_2 \)-module.

(\textit{Hint.} Consider the finite-dimensional algebra \( A \) that is the image of \( U(g) \) under the induced algebra homomorphism \( \rho : U(g) \to \text{End}_C(V) \), use Wedderburn theorem.)

2. Let \( g \) be a finite-dimensional semisimple Lie algebra with Killing form \( \kappa \). We showed in L6-2 that \( g \) is the direct sum \( g_1 \oplus \cdots \oplus g_n \) of its simple ideals, with \( g_i \perp \kappa g_j \) for \( i \neq j \). Let \( \beta \) be some other invariant bilinear form on \( g \). Prove:

(a) \( \beta |_{g_i} \) is a multiple of the Killing form \( \kappa |_{g_i} \) for each \( i = 1, \ldots, n \).
(b) \( g_i \perp \beta g_i \) for \( i \neq j \).
(c) \( \beta \) is a symmetric bilinear form.

The remaining questions require the notion of a \emph{toral subalgebra} \( t \) of a finite-dimensional semisimple Lie algebra \( g \). Recall that this is a subalgebra consisting entirely of semisimple elements of \( g \). As established in L7-1, such a subalgebra is necessarily Abelian. Hence, there is a \emph{root space decomposition}

\[
g = g_0 \oplus \bigoplus_{\alpha \in R} g_{\alpha}
\]

where \( R = \{0 \neq \alpha \in t^* \mid g_{\alpha} \neq 0\} \) is the set of \emph{roots} of \( g \) with respect to \( t \). Here,

\[
g_{\alpha} = \{x \in t \mid [t,x] = \alpha(t)x \text{ for all } t \in t\}
\]

for any \( \alpha \in t^* \).

3. Suppose that \( t \) is a toral subalgebra of a finite-dimensional semisimple Lie algebra \( g \) such that \( t \) is equal to its centralizer \( c_g(t) \). Why does this imply that \( t \) is actually a \emph{maximal} toral subalgebra of \( g \)? Use this observation to show that the subalgebra of \( \text{sl}_n(\mathbb{C}) \) consisting of all diagonal matrices of trace zero is a maximal toral subalgebra.
Now let $g = \mathfrak{sp}_{2n}(\mathbb{C})$ using the coordinates chosen in L3-2. Let $t$ be the subalgebra of all diagonal matrices in $g$, and note that any element $t \in t$ can be written as $t = \text{diag}(t_1, \ldots, t_n, -t_n, \ldots, -t_1)$ for $t_1, \ldots, t_n \in \mathbb{C}$. Let $\varepsilon_i \in t^*$ be the linear map sending $t \mapsto t_i$ for $i = 1, \ldots, n$, so that $\varepsilon_1, \ldots, \varepsilon_n$ give a basis for $t^*$.

4. Describe the root space decomposition of $g = \mathfrak{sp}_{2n}(\mathbb{C})$ with respect to $t$ using the notation just introduced. You should show in particular that

$$R = \{ \pm \varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_k \mid 1 \leq i < j \leq n, 1 \leq k \leq n \},$$

that $g_\alpha$ is one-dimensional for each $\alpha \in R$, and that $g_0 = t$. Deduce that $t$ is a maximal toral subalgebra.

5. Let $V = \mathbb{C}^{2n}$ be the natural representation of $g = \mathfrak{sp}_{2n}(\mathbb{C})$ denoting its standard ordered basis $v_1, \ldots, v_n, v_{-n}, \ldots, v_{-1}$.

(a) We noted in L5-3 that $V$ is an irreducible $g$-module. Convince yourself of this again! It follows that $g$ is indeed a semisimple Lie algebra.

(b) Show that $v_i$ is of weight $\varepsilon_i$ and $v_{-i}$ is of weight $-\varepsilon_i$, where $\varepsilon_1, \ldots, \varepsilon_n \in t^*$ are as defined above.

(c) Describe how $S^2(V)$ decomposes into weight spaces with respect to $t$. Then show that $S^2(V) \cong g$ (the adjoint representation) as $g$-modules.

(d) Show that $S^2(V)$ is an irreducible $g$-module. Deduce that $g$ is a simple Lie algebra.

(Hint. For (a) and/or (d) you might find it easier to prove irreducibility as representations of the algebraic group $G = Sp_{2n}(\mathbb{C})$.)