1. Let $A$ be a finite-dimensional algebra (associative or Lie or whatever) over an algebraically closed field $k$. Let $d : A \to A$ be a derivation. Show that the semisimple and nilpotent parts $d_s$ and $d_n$ of $d \in \text{End}_k(A)$ are also derivations of $A$.

(*Hint.* It suffices to prove that $d_s \in \text{Der}(A)$. Decompose $A$ into generalized eigenspaces $A = \bigoplus_{\lambda \in k} A_\lambda$ for $d$, so that $d_s$ acts on $A_\lambda$ by multiplication by $\lambda$, then show that $A_\lambda A_\mu \subseteq A_{\lambda+\mu}$.)

2. Recall for a Lie algebra $\mathfrak{g}$ that $\text{Der}(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$.

   (a) Show that $\text{ad}(\mathfrak{g})$ is an ideal of $\text{Der}(\mathfrak{g})$, the so-called *inner derivations*.

   Now assume $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra over $\mathbb{C}$.

   (b) Given a derivation $d : \mathfrak{g} \to \mathfrak{g}$, show that the vector space $\mathfrak{g} \oplus \mathbb{C}$ is a well-defined $\mathfrak{g}$-module with action $x(y, \lambda) = ([x, y] + \lambda d(x), 0)$.

   (c) Use Weyl’s theorem on complete reducibility to prove that every derivation of $\mathfrak{g}$ is inner, i.e., $\text{Der}(\mathfrak{g}) = \text{ad}(\mathfrak{g})$.

3. For a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ and $x \in \mathfrak{g}$, we showed in L6-3 that there are unique elements $x_s, x_n \in \mathfrak{g}$ such that $x = x_s + x_n$, $[x_s, x_n] = 0$, $\text{ad} x_s : \mathfrak{g} \to \mathfrak{g}$ is semisimple and $\text{ad} x_n : \mathfrak{g} \to \mathfrak{g}$ is nilpotent. Use Q1 and Q2 to give another proof of this.

4. This question is the beginning of *Lie algebra cohomology*. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$.

   (a) Suppose that there is a Lie algebra extension

   $$0 \to \mathfrak{g} \to \hat{\mathfrak{g}} \to \mathbb{C} \to 0,$$

   i.e., $\hat{\mathfrak{g}}$ is a Lie algebra containing $\mathfrak{g}$ as an ideal of codimension one. Prove that $\hat{\mathfrak{g}} \cong \mathfrak{g} \oplus \mathbb{C}$, i.e., it is a split extension.

   (b) Suppose that there is a Lie algebra extension

   $$0 \to \mathbb{C} \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0.$$
Show that the first map embeds \( \mathbb{C} \) into the center \( \mathfrak{z}(\hat{\mathfrak{g}}) \), i.e., it is a central extension. Then prove that the extension is split.

(Hints. For (a), the data of such an extension is really just the data of a derivation \( d \) of \( \mathfrak{g} \); then you can use Q2. Part (b) is more difficult!)

5. The infinite-dimensional Lie algebra constructed in the next question is the affine Lie algebra \( \hat{\mathfrak{sl}}_2(\mathbb{C}) \). Let \( \mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \). Let \( \mathfrak{g}[t, t^{-1}] \) be the infinite-dimensional Lie algebra \( \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \) with \([xt^m, yt^n] = [xy]t^{m+n}\) for \( x, y \in \mathfrak{g}, m, n \in \mathbb{Z} \).

(a) Show that there is a non-split Lie algebra extension

\[
0 \rightarrow \mathbb{C} \rightarrow \hat{\mathfrak{g}}' \rightarrow \mathfrak{g}[t, t^{-1}] \rightarrow 0
\]

such that \( \hat{\mathfrak{g}}' = \mathbb{C} c \oplus \mathfrak{g}[t, t^{-1}] \) as a vector space with Lie bracket defined so that \( c \) is central and

\[
[xt^m, yt^n] = [x, y]t^{m+n} + \delta_{m+n,0}\tau(x, y)mc,
\]

where \( \tau \) is the trace form on \( \mathfrak{sl}_2(\mathbb{C}) \).

(b) Let \( \hat{\mathfrak{g}}' \) be as in (a). Show that there is a non-split Lie algebra extension

\[
0 \rightarrow \hat{\mathfrak{g}}' \rightarrow \hat{\mathfrak{g}} \rightarrow \mathbb{C} \rightarrow 0
\]

such that \( \hat{\mathfrak{g}} = \hat{\mathfrak{g}}' \oplus \mathbb{C} d \) as a vector space with the Lie bracket defined so that \( \hat{\mathfrak{g}}' \) is a Lie subalgebra, \([d, c] = 0\) and \([d, xt^n] = nx t^n\).

(c) Show that \( \mathfrak{z}(\hat{\mathfrak{g}}) = \mathbb{C} c \) and that \( \hat{\mathfrak{g}}' \) is the derived subalgebra of \( \hat{\mathfrak{g}} \).