Introduction to Lie Theory
Homework #7

1. Let A be a finite-dimensional algebra (associative or Lie or whatever) over an algebraically closed field $\Bbbk$. Let $d : A \to A$ be a derivation. Show that the semisimple and nilpotent parts d_s and d_n of $d \in \mathrm{End}_{\Bbbk}(A)$ are also derivations of A.

   (*Hint.* It suffices to prove that $d_s \in \mathrm{Der}(A)$. Decompose A into generalized eigenspaces $A = \bigoplus_{\lambda\in\Bbbk} A_\lambda$ for d, so that d_s acts on A_λ by multiplication by λ, then show that $A_\lambda A_\mu \subseteq A_{\lambda+\mu}$.)

2. Recall for a Lie algebra $\mathfrak{g}$ that $\mathrm{Der}(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$.

   (a) Show that $\mathrm{ad}(\mathfrak{g})$ is an ideal of $\mathrm{Der}(\mathfrak{g})$, the so-called *inner derivations.*

   Now assume $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra over $\mathbb{C}$.

   (b) Given a derivation $d : \mathfrak{g} \to \mathfrak{g}$, show that the vector space $\mathfrak{g} \oplus \mathbb{C}$ is a well-defined $\mathfrak{g}$-module with action $x(y, \lambda) = ([x, y] + \lambda d(x), 0)$.

   (c) Use Weyl's theorem on complete reducibility to prove that every derivation of $\mathfrak{g}$ is inner, i.e., $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$.

3. For a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ and $x \in \mathfrak{g}$, we showed in L6-3 that there are unique elements $x_s, x_n \in \mathfrak{g}$ such that $x = x_s + x_n, [x_s, x_n] = 0$, $\mathrm{ad}\, x_s : \mathfrak{g} \to \mathfrak{g}$ is semisimple and $\mathrm{ad}\, x_n : \mathfrak{g} \to \mathfrak{g}$ is nilpotent. Use Q1 and Q2 to give another proof of this.

4. This question is the beginning of *Lie algebra cohomology.* Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$.

   (a) Suppose that there is a Lie algebra extension

   $$0 \to \mathfrak{g} \to \widehat{\mathfrak{g}} \to \mathbb{C} \to 0,$$

   i.e., $\widehat{\mathfrak{g}}$ is a Lie algebra containing $\mathfrak{g}$ as an ideal of codimension one. Prove that $\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus \mathbb{C}$, i.e., it is a split extension.

   (b) Suppose that there is a Lie algebra extension

   $$0 \to \mathbb{C} \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0.$$

Show that the first map embeds $\mathbb{C}$ into the center $\mathfrak{z}(\widehat{\mathfrak{g}})$, i.e., it is a *central extension*. Then prove that the extension is split.

(*Hints.* For (a), the data of such an extension is really just the data of a derivation d of $\mathfrak{g}$; then you can use Q2. Part (b) is more difficult!)

5. The infinite-dimensional Lie algebra constructed in the next question is the *affine Lie algebra* $\widehat{\mathfrak{sl}}_2(\mathbb{C})$. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Let $\mathfrak{g}[t, t^{-1}]$ be the infinite-dimensional Lie algebra $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ with $[xt^m, yt^n] = [xy]t^{m+n}$ for $x, y \in \mathfrak{g}, m, n \in \mathbb{Z}$.

   (a) Show that there is a non-split Lie algebra extension

   $$0 \to \mathbb{C} \to \widehat{\mathfrak{g}}' \to \mathfrak{g}[t, t^{-1}] \to 0$$

   such that $\widehat{\mathfrak{g}}' = \mathbb{C}c \oplus \mathfrak{g}[t, t^{-1}]$ as a vector space with Lie bracket defined so that c is central and

   $$[xt^m, yt^n] = [x, y]t^{m+n} + \delta_{m+n,0}\tau(x, y)mc,$$

   where τ is the trace form on $\mathfrak{sl}_2(\mathbb{C})$.

   (b) Let $\widehat{\mathfrak{g}}'$ be as in (a). Show that there is a non-split Lie algebra extension

   $$0 \to \widehat{\mathfrak{g}}' \to \widehat{\mathfrak{g}} \to \mathbb{C} \to 0$$

   such that $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}' \oplus \mathbb{C}d$ as a vector space with the Lie bracket defined so that $\widehat{\mathfrak{g}}'$ is a Lie subalgebra, $[d, c] = 0$ and $[d, xt^n] = nxt^n$.

   (c) Show that $\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}c$ and that $\widehat{\mathfrak{g}}'$ is the derived subalgebra of $\widehat{\mathfrak{g}}$.