Introduction to Lie Theory
Homework #8

1. Suppose that E is a Euclidean space with basis α, β such that

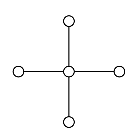
$$(\alpha, \beta^\vee) = -1, \qquad\qquad (\beta, \alpha^\vee) = -3.$$

where $\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)}$ and $\beta^\vee := \frac{2\beta}{(\beta,\beta)}$ as usual. Suppose that $R \subset E$ is a root system with $\alpha, \beta \in R$. Prove that $|R| = 12$, indeed, it is the root system of type G_2 with Cartan matrix $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ with respect to the base $\Delta = \{\alpha, \beta\}$.

2. I did not prove the classification of root systems in the lectures. This question should give you some idea about how you might go about eliminating possibilities for the Dynkin diagrams by "finding enough obstructions."

   (a) Suppose that $R \subset E$ is a root system with base $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$. Let Δ' be a non-empty subset of Δ, E' be the Euclidean subspace of E spanned by Δ', and $R' = R \cap E'$. Prove that $R' \subset E'$ is a root system with base Δ'.

   (b) Show that the graph



   is not the Dynkin diagram of a root system. (*Hint.* What is the determinant of the corresponding Cartan matrix?)

   (c) Deduce that all vertices in the Dynkin diagram of a root system in which all roots of the same length are of degree ≤ 3.

3. Let $W < GL(E)$ be the Weyl group of a root system $R \subset E$. Let $s_1, \ldots, s_\ell \in W$ be the simple reflections arising from a choice of a base $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ for R. Prove that the determinant of $w \in W$ viewed as a linear map $w : E \to E$ is equal to $(-1)^n$ for any (not necessarily reduced) expression $w = s_{i_1} \cdots s_{i_n}$ as a product of n simple

reflections. Deduce that $n \equiv \ell(w) \pmod 2$ where $\ell : W \to \mathbb{N}$ is the length function, and that the map $\mathrm{sgn} : W \to \{\pm 1\}, w \mapsto (-1)^{\ell(w)}$ is a group homomorphism.

4. Let $R \subset E$ be a root system with base Δ. Recall that the *height* of a root is the sum of its coefficients when it is written as a linear combination of the base. Prove that there is a unique *highest root* $\alpha_0 \in R^+$. In fact, α_0 is the unique element $\alpha_0 = \sum_{i=1}^{\ell} a_i \alpha_i \in R^+$ such that $a_i \geq b_i$ for every $\beta = \sum_{i=1}^{\ell} b_i \alpha_i \in R^+$. Moreover, in the case that R is indecomposable with two different root lengths, show that α_0 is one of the long roots.

   (*Hints.* It suffices to treat the case that R is indecomposable. Suppose that $\alpha = \sum_{i=1}^{\ell} a_i \alpha_i$ is a positive root of maximal height. Using the first lemma from L8-1, you deduce that $(\alpha, \alpha_i) \geq 0$ for all $i = 1, \dots, \ell$ and that $(\alpha, \alpha_i) = 0$ whenever $a_i = 0$. By considering the partition $\Delta = \Delta_1 \sqcup \Delta_2$ where $\Delta_1 = \{\alpha_i \,|\, a_i > 0\}$ and $\Delta_2 = \{\alpha_i \,|\, a_i = 0\}$ and using the indecomposability, you see if $\Delta_2 \neq \varnothing$ that there must be $\alpha_i \in \Delta_1$ and $\alpha_j \in \Delta_2$ with $(\alpha_i, \alpha_j) < 0$. This contradicts $(\alpha, \alpha_j) \geq 0$. Hence, $a_i > 0$ for all i. Then take another positive root β of maximal height and show that $(\alpha, \beta) > 0$, so $\alpha - \beta$ is a root, which is a contradiction since it must have some positive and some negative coefficients.)

5. Prove that the Weyl group of a root system has a unique *longest* element w_0. Moreover, w_0 is the reflection s_{α_0} in the hypoerplane orthogonal to the highest root α_0 from the previous question, hence, $w_0^2 = 1$. Show also that $\ell(w_0) = |R^+|$. Is $w_0 \in Z(W)$?

6. Suppose that $\lambda = \sum_{i=1}^{\ell} c_i \alpha_i$ for $c_i \in \mathbb{N}$. Show that λ is a multiple of a root if and only if $w(\lambda)$ is either a positive sum of simple roots or a negative sum of simple roots for all $w \in W$.

   (*Hint.* Suppose that λ is not a multiple of any root. Take $\mu \in E - \bigcup_{\alpha \in R} \alpha^\perp$ lying on the hyperplane $\lambda^\perp$. Then choose $w \in W$ so that $w(\mu)$ lies in the fundamental chamber, i.e., $(\alpha_i, w(\mu)) > 0$ for all $i \in I$. Finally consider $w(\lambda)$, noting that $(w(\lambda), w(\mu)) = 0$.)