Ch. 0 Why Lie algebras?

Always algebras mean $\mathbb{K}$-algebras for some field $\mathbb{K}$, usually $\mathbb{K}=\mathbb{C}$

Main topic: finite-dimensional semisimple Lie algebras / $\mathbb{C}$

Example Let A be any associative algebra xy

Define $[x,y] = xy - yx$. "commutator"

This makes A into a Lie algebra

Anti-symmetry ✓ Jacobi?

- $[x, yz] = [x,y]z + y[x,z]$
- $[x,[y,z]] = [[x,y]z] + [y[x,z]]$

The latter rewrites to give Jacobi.

Says that $D_x : A \to A$, $a \mapsto [x,a]$ is a derivation of assoc. algebra A (Leibniz / product rule)

A Lie algebra is a vectorspace $\mathfrak{g}$ plus a bilinear map

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying

1. (anti-symmetry) $[x,x] = 0 \quad \forall x \in \mathfrak{g}$ $\Rightarrow$ $[x,y] = -[y,x]$
2. (Jacobi identity) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \quad \forall x,y,z \in \mathfrak{g}$

Says that $D_x : \mathfrak{g} \to \mathfrak{g}$, $a \mapsto [x,a]$ is a derivation of Lie algebra $\mathfrak{g}$.

For example, take $A = M_n(\mathbb{k})$ ($n \times n$ matrices under matrix mult.)

Then the above construction turning A into a Lie algebra produces the Lie algebra $\mathfrak{gl}_n(\mathbb{k})$, the <u>general linear Lie algebra</u>.

The definition of Lie algebra probably seems unmotivated!!

I'm going to spend 2 weeks talking instead about

algebraic groups

with my goal being to explain why Lie algebras??

<u>Main idea of Lie theory</u>

• Lie algebras are easy

• There's a very tight connection between structure of a connected algebraic group/$\mathbb{C}$ and its Lie algebra

linearized groups

Put your favorite adjective here!

undergraduate : finite

topologist : compact Lie, eg S^1, SO_3

analyst : locally compact, $(\mathbb{R},+)$

algebraist : me!!

Why <u>Lie algebras</u>? <u>Algebraic groups</u>.

Now I'm going to define algebraic groups. Much more difficult as the definition rests on some language from <u>algebraic geometry</u>.

just need <u>affine varieties</u>

summary of these !!

Background from algebraic geometry

- $\mathbb{k} = \overline{\mathbb{k}}$ (usually $\mathbb{k} = \mathbb{C}$)
- $\boxed{\text{affine variety}}$ X is a set plus a specified <u>finitely generated</u> algebra $\mathbb{k}[X]$ <u>of functions</u> from X to $\mathbb{k}$, plus an <u>axiom</u>...

coordinate algebra of X

Means: $\mathbb{k}[T_1, \dots, T_n] \twoheadrightarrow \mathbb{k}[X]$ some n

$\therefore$ By Hilbert basis theorem, get that $\mathbb{k}[X]$ is Noetherian.

Means: $\mathbb{k}[X]$ is subalgebra of $\mathrm{Map}(X, \mathbb{k})$

commutative

automatically <u>reduced</u>

(no non-zero nilpotent elements)

ALL functions $X \to \mathbb{k}$

under pointwise operations

(commutative)

- If X is an affine variety, have <u>Zariski topology</u> on X in which the closed sets are the sets

"vanishing set of I"

$$V(I) = \{ x \in X \mid f(x) = 0 \ \forall f \in I \}$$

for $I \trianglelefteq \Bbbk[X]$

If $I = (f_1, \dots, f_n)$ then $V(I) = V(f_1) \cap \dots \cap V(f_n)$

where $V(f) =$ "$f(x) = 0$"

eg $\mathbb{A}^n$ affine n-space

Set $\Bbbk^n$ of n-tuples of elements of $\Bbbk$

irreducible

$\Bbbk[\mathbb{A}^n] = \Bbbk[T_1, \dots, T_n]$

T_i = ith coordinate function

$(f \in \Bbbk[X])$

Let $D(f) = X - V(f)$ "$f(x) \neq 0$" <u>principal open subset</u>

Note $D(f)$ is open, and any open set is a <u>finite</u> union of principal opens.

In particular, $D(f)$'s are a base for the Zariski topology.

- Call X <u>irreducible</u> if cannot write X as $X = X_1 \cup X_2$ for proper closed subsets X_1, X_2 $\iff$ $\Bbbk[X]$ is an integral domain.

- <u>Nullstellensatz</u> Note $V(f) = V(f^2) = \dots$ so $V(I) = V(\sqrt{I})$

a <u>radical</u> ideal

$\mathbb{k}[X]/\sqrt{I}$ is reduced

$$\left\{ \begin{matrix} \text{radical ideals} \\ I \trianglelefteq \mathbb{k}[X] \end{matrix} \right\} \underset{I}{\overset{V}{\rightleftarrows}} \left\{ \begin{matrix} \text{closed sets in } X \\ \text{wrt Z.T.} \end{matrix} \right\}$$

mutually inverse, inclusion reversing bijections.

For $Y \subseteq X$, $I(Y) = \{ f \in \mathbb{k}[X] \mid f(y) = 0 \ \forall y \in Y \}$.

In particular:

$$\left\{ \begin{matrix} \text{maximal ideals} \\ M \triangleleft \mathbb{k}[X] \end{matrix} \right\} \overset{\sim}{\longleftrightarrow} X \qquad V(M_x) = \{x\}$$

$$M_x = \ker ev_x \longleftarrow\!\shortmid \ x$$

$ev_x : \mathbb{k}[X] \twoheadrightarrow \mathbb{k}$

"evaluation at x"

In fact, that this map is a bijection is the missing axiom for "affine variety".

- $\varphi : X \to Y$ morphism of affine varieties

A function such that $\varphi^* : \mathrm{Map}(Y, \mathbb{k}) \to \mathrm{Map}(X, \mathbb{k})$

$f \mapsto f \circ \varphi$

takes $\mathbb{k}[Y]$ into $\mathbb{k}[X]$.

$\varphi^* : \mathbb{k}[Y] \to \mathbb{k}[X]$

Automatically cts for Z.T.

comorphism of φ

You can recover φ from φ^*

eg $ev_x : \mathbb{k}[X] \to \mathbb{k}$

Indeed, given any $\Theta : \mathbb{k}[Y] \to \mathbb{k}[X]$, alg. hom,

is the comorphism of

there's a unique $\varphi : X \to Y$ with $\varphi^* = \Theta$.

$inc_x : \{x\} \hookrightarrow X$

"inclusion of point" $x \in X$

For $x \in X$, what's $\varphi(x) \in Y$?

$inc_{\varphi(x)} = \varphi \circ inc_x$

$ev_{\varphi(x)} = ev_x \circ \cancel{\varphi^*}\ \Theta$

$ev_x \circ \Theta : \mathbb{k}[Y] \twoheadrightarrow \mathbb{k}$

$\ker ev_x \circ \Theta = M_y$ for $!\, y \in Y$

max-ideal

" $\ker ev_y$ that y is $\varphi(x)$.