- Affine variety, coordinate algebra, Zariski topology, irreducible variety
- Morphisms, isomorphisms, $\mathbb{A}^n$, Nullstellensatz.
- <u>Construction of new affine varieties out of old</u>

① If Y is a closed subset of affine variety X, you make Y into an affine variety by setting $\mathbb{k}[Y] := \{ f|_Y \mid f \in \mathbb{k}[X] \} \cong \mathbb{k}[X]/I(Y)$

restriction defines a surjective algebra hom. from $\mathbb{k}[X]$ to $\mathbb{k}[Y]$, with kernel $I(Y)$

Then inclusion $\varphi : Y \hookrightarrow X$ is a morphism of affine varieties with φ^* being that restriction homomorphism.

Such an inclusion is called a <u>closed embedding</u> $\longleftrightarrow$ $\varphi^* : \mathbb{k}[X] \twoheadrightarrow \mathbb{k}[Y]$ <u>surjective algebra hom.</u>

eg $\mathbb{k}[T_1, \dots, T_n] \twoheadrightarrow \mathbb{k}[X]$, $T_i \mapsto f_i$ $\quad$ choice of generators $f_1, \dots, f_n$ $\quad$ Gives a choice of <u>coordinates</u> for X

$\updownarrow$ $\qquad\qquad$ $\updownarrow$

$X \hookrightarrow \mathbb{A}^n$ $\quad$ closed embedding of X into $\mathbb{A}^n$

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow & & \downarrow \\ \mathbb{A}^n & \longrightarrow & \mathbb{A}^m \end{array}$$

morphism of affine varieties

In coordinates, for $x = (x_1, \dots, x_n) \in X \subseteq \mathbb{A}^n$,

$$\varphi(x) = (\varphi_1(x), \dots, \varphi_m(x)) \in Y \subseteq \mathbb{A}^m$$

Each φ_i is a polynomial in coordinates $x_1, \dots, x_n$ of x.

A morphism is a polynomial function when expressed in coordinates.

② X, Y two affines. $X \times Y$ (Cartesian product) is an affine variety with $\mathbb{k}[X \times Y] := \mathbb{k}[X] \otimes_{\mathbb{k}} \mathbb{k}[Y]$

$$\begin{cases} (f \otimes g)((x,y)) = f(x) \cdot g(y) \\ M_{(x,y)} = M_x \otimes \mathbb{k}[Y] + \mathbb{k}[X] \otimes M_y \end{cases}$$

This gives product in category of affine varieties in general sense.

$$\mathbb{A}^n = \underbrace{\mathbb{A}^1 \times \dots \times \mathbb{A}^1}_{n}$$

③ X affine, $f \in \mathbb{k}[X]$ $D(f)$ "$f(x) \neq 0$" principal open subset

$D(f)$ is itself an affine variety with $\mathbb{k}[D(f)] := \mathbb{k}[X]_f$

localisation at mult set $\{1, f, f^2, \cdots\}$

$\left\{ \frac{g}{f^n} \;\middle|\; g \in \mathbb{k}[X], n \geq 0 \right\}$

How do you prove the axiom? $\left(\frac{g}{f^n}\right)(x) = \frac{g(x)}{(f(x))^n}$

$x \in D(f)$

$f(x) \neq 0$

Consider the closed subset

$$V(fT-1) \subset X \times \mathbb{A}^1 \longleftrightarrow \mathbb{k}[X] \otimes \mathbb{k}[T] = \mathbb{k}[X][T]$$

Its affine variety by ①, $\mathbb{k}[V(fT-1)] = \mathbb{k}[X][T] / (fT-1)$

$$\frac{\quad}{g\,T^n} \qquad \Big\updownarrow \cong$$

$$\frac{g}{f^n} \qquad \mathbb{k}[X]_f$$

This is isomorphism of

$$D(f) \xrightarrow{\sim} V(fT-1)$$

$$x \longmapsto \left(x, \frac{1}{f(x)}\right)$$

Idea: $\overline{fT} = 1$

$\overline{T} = \frac{1}{f}$.

- **Dimension** Say X is an irreducible affine variety $\iff \mathbb{k}[X]$ integral dom.

Let $\mathbb{k}(X)$ be the function field of X, the field of fractions of $\mathbb{k}[X]$.

Then $\dim X := \operatorname{trdeg.} \mathbb{k}(X)/\mathbb{k}$

eg $\dim \mathbb{A}^n = n \leftrightarrow \operatorname{trdeg.} \mathbb{k}(T_1, \dots, T_n)/\mathbb{k} = n$

eg $0 \neq f \in X$, then $D(f)$ is irreducible, and $\dim D(f) = \dim X$

as $\mathbb{k}[D(f)] = \mathbb{k}[X]_f$ so $\mathbb{k}(D(f)) = \mathbb{k}(X)$

Th. $\dim X = \max \{ n \in \mathbb{N} \mid \exists\, X = X_1 \supsetneq X_2 \supsetneq \dots \supsetneq X_n \neq \emptyset \}$

all X_i are irreducible closed in X

Definition An $\boxed{\text{algebraic group}}$ is an affine variety G with structure of a group so that group multiplication $m: G \times G \to G,\ (g,h) \mapsto gh$ and inversion $i: G \to G,\ g \mapsto g^{-1}$ are both morphisms of varieties.

A <u>morphism of algebraic groups</u> is a morphism of varieties $\Phi: G \to H$ that is also a group homomorphism.

eg $GL_n(\mathbb{k})$

$GL_n(\mathbb{k}) = D(\det)$

Matrices with non-zero determinant

$\mathbb{k}[GL_n(\mathbb{k})] = \mathbb{k}[T_{ij}]_{\det}$.

$M_n(\mathbb{k}) \cong \mathbb{A}^{n^2}$

$n \times n$ matrices

affine variety!

ij-coordinate function

$\mathbb{k}[M_n(\mathbb{k})] = \mathbb{k}[T_{ij} \mid 1 \leq i,j \leq n]$

$\det = \det \begin{bmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nn} \end{bmatrix}$

It's also a group, m is obviously a morphism (quadratic in matrix entries), as is i.

Let G be an algebraic group. $\qquad \mathbb{k}[G]$ coordinate algebra

$m: G\times G \to G \quad \longleftrightarrow \quad m^*: \mathbb{k}[G] \to \mathbb{k}[G\times G] = \mathbb{k}[G]\otimes\mathbb{k}[G]$ comultiplication

$i: G \to G \quad \longleftrightarrow \quad i^*: \mathbb{k}[G] \to \mathbb{k}[G]$ antipode

$\mathrm{inc}_e: \{e\} \to G \quad \longleftrightarrow \quad \mathrm{inc}_e^* = \mathrm{ev}_e : \mathbb{k}[G] \to \mathbb{k},\ f \mapsto f(e)$ counit

algebra homomorphism.

Take group axioms and translate in terms of these comorphism.

① associativity $\longleftrightarrow$ coassociativity

$$\begin{tikzcd}
\mathbb{k}[G] \arrow[r, "m^*"] \arrow[d, "m^*"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{k}[G]\otimes\mathbb{k}[G] \arrow[d, "m^*\otimes \mathrm{id}"] \\
\mathbb{k}[G]\otimes\mathbb{k}[G] \arrow[r, "\mathrm{id}\otimes m^*"'] & \mathbb{k}[G]\otimes\mathbb{k}[G]\otimes\mathbb{k}[G]
\end{tikzcd}$$

This diagram commutes.

Coassociativity.

② $eg = g = ge \quad \forall g \in G$

$$\begin{tikzcd}
 & G \times G \arrow[d, "m"] \\
\{e\} \times G \arrow[ur, "\mathrm{incl}_e \times \mathrm{id}"] \arrow[r] \arrow[ur, phantom, "\circ", shift left=4] & G
\end{tikzcd}$$

$$(e,g) \mapsto g$$

$$\begin{tikzcd}
 & \Bbbk[G] \otimes \Bbbk[G] \arrow[dl, "ev_e \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id} \otimes ev_e"] & \\
\Bbbk \otimes \Bbbk[G] \arrow[dr, equal, "\mathrm{can}"'] \arrow[r, phantom, "\circ"] & {} & \Bbbk[G] \otimes \Bbbk \arrow[dl, equal, "\mathrm{can}"] \arrow[l, phantom, "\circ"] \\
 & \Bbbk[G] \arrow[uu, "m^*"] &
\end{tikzcd}$$

This commutes.

Counital property.

③ $gg^{-1} = e = g^{-1}g \quad \forall g \in G$

$$f\left[m\left((\mathrm{id}\times i)(\delta(g))\right)\right] = f\left[\mathrm{inc}_e(p(g))\right] \quad \forall g \in G \quad \forall f \in \mathbb{k}[G]$$

$$\Updownarrow$$

$$\left[\left(\delta^* \circ (\mathrm{id}\otimes i^*) \circ m^*\right)(\cancel{f})\right](\cancel{g}) = \left[\left(p^* \circ ev_e\right)(\cancel{f})\right](\cancel{g}) \quad \forall \cancel{g}, \cancel{f}$$

$$\Updownarrow$$

$$\begin{tikzcd}
\mathbb{k}[G]\otimes\mathbb{k}[G] \arrow[dd, "i^*\otimes\mathrm{id}"'] \arrow[ddr, phantom, "\circ"] & \mathbb{k}[G] \arrow[l, "m^*"'] \arrow[r, "m^*"] \arrow[d, "ev_e"] \arrow[ddr, phantom, "\circ"] & \mathbb{k}[G]\otimes\mathbb{k}[G] \arrow[dd, "\mathrm{id}\otimes i^*"] \\
 & \mathbb{k} \arrow[d, "\mathrm{can}"] & \\
\mathbb{k}[G]\otimes\mathbb{k}[G] \arrow[r, "\mathrm{mult}"] & \mathbb{k}[G] & \mathbb{k}[G]\otimes\mathbb{k}[G] \arrow[l, "\mathrm{mult}"']
\end{tikzcd}$$

$\delta : G \to G\times G$

$g \mapsto (g,g)$

"diagonal"

$\delta^* : \mathbb{k}[G]\otimes\mathbb{k}[G] \to \mathbb{k}[G]$

alg. multiplication!

$p : G \to \{e\}$

$g \mapsto e$

$p^* : \mathbb{k} \to \mathbb{k}[G]$

$1 \mapsto 1$

algebra unit

antipode property.

<u>Def</u>: An associative, unital algebra A is called a <u>bialgebra</u> if there's an algebra hom. $m^*: A \to A \otimes_{\mathbb{k}} A$ and $ev_e : A \to \mathbb{k}$

given so diagrams ① & ② commute.

If in addition there's a linear map $i^*: A \to A$ so ③ commutes, the bialgebra is called a <u>Hopf algebra</u>.

(In that case i^* is unique ... property not new data, and i^* is an algebra antihomomorphism for free.)

So: $\mathbb{k}[G]$ is a commutative Hopf algebra.