There are a few loose ends ... I want to expand these in the last three lectures.

PLAN

(1) A glimpse of representation theory

So I can discuss Weyl dimension formula which is needed in proof of Lemma from L9-2.

(2) More about exponentiating

The cunning trick in the proof of Serre's theorem' in L9-3. Needs more discussion!

(3) Construction of semisimple algebraic groups starting from semisimple Lie algebras.

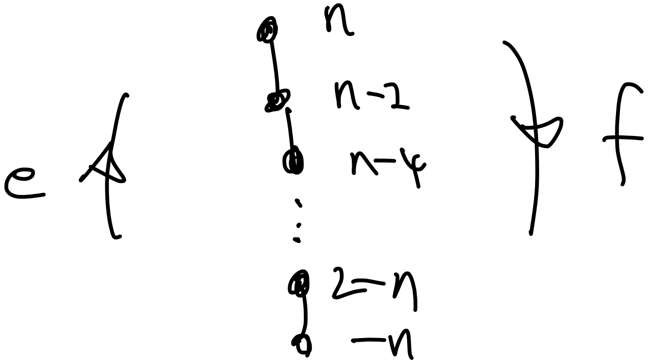
A glimpse of representation theory

EXTEND IT ALL TO GENERAL $\mathfrak{g}$

We developed $\mathrm{Rep}(\mathfrak{sl}_2(\mathbb{C}))$ in detail already:

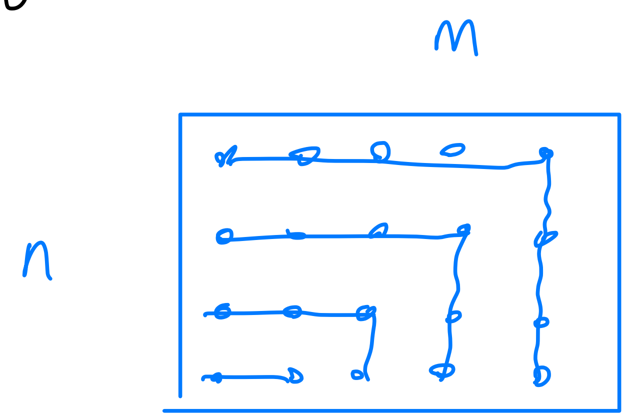
- Every f.d. $\mathfrak{sl}_2(\mathbb{C})$-module is c.r.
- Irreducible ones $/\cong$ are labelled by their highest weights $n \in \mathbb{N}$

$$L(n) = S^n V \quad \text{where } V = \mathbb{C}^2 \text{ is natural 2-D rep.}$$



Labels are h-eigenvalue of a basis

weights



- $\dim L(n) = n+1$
- h acts diagonalizably on any f.d. rep.
- $$L(n) \otimes L(m) \cong \bigoplus_{\substack{p=|n-m| \\ p \equiv n+m \ (\mathrm{mod}\ 2)}}^{n+m} L(p)$$

Clebsch-Gordan formula

HW 5-Q1

Let $\mathfrak{g}$ be a f.d. semisimple Lie algebra ... $\mathfrak{g} = \mathfrak{g}(C)$ ← a Cartan matrix

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ //

$\Delta = \{\alpha_1, \ldots, \alpha_\ell\} \subset R \subset E$ root system

base

$\langle e_i, h_i, f_i \mid i \in I \rangle$

Serre presentation

$\mathfrak{t} = \langle h_1, \ldots, h_\ell \rangle$ max. toral subalgebra

$$\mathfrak{g} = \underbrace{\mathfrak{n}^-}_{\langle f_i \rangle} \oplus \underbrace{\mathfrak{t}}_{\langle h_i \rangle} \oplus \underbrace{\mathfrak{n}^+}_{\langle e_i \rangle}$$

$\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}^+$ Borel subalgebra

corresponding to $R^+ \subset R$.

We know already:

- Any element $h \in \mathfrak{t}$ is semisimple, hence, it acts semisimply on $\underline{\text{any}}$ f.d. representation of $\mathfrak{g}$ (Jordan decomposition). As $\mathfrak{t}$ is Abelian any $V \in \operatorname{Rep}(\mathfrak{g})$ decomposes as $$V = \bigoplus_{\lambda \in \mathfrak{t}^*} \underbrace{V_\lambda}_{\{v \in V \mid hv = \lambda(h) v \ \forall h \in \mathfrak{t}\}}$$
- Any f.d. representation is c.r.

Def $\lambda \in \mathfrak{h}^*$. A highest weight module of h/w λ is a $\mathfrak{g}$-module generated by a non-zero highest weight vector v_+ of weight λ.

$$e_i v_+ = 0, \quad h_i v_+ = \lambda(h_i) v_+ \quad \forall i = 1, \dots, \ell$$

$$\Updownarrow$$

$$\mathfrak{n}^+ v_+ = 0$$
$$h v_+ = \lambda(h) v_+ \quad \forall h \in \mathfrak{h}.$$

There's a universal such module, the Verma module ← ∞-D

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$$

$v_+ = 1 \otimes 1$ h/w vector

1-D $\mathfrak{b}$-module generated by a vector satisfying all that stuff.

$\{ f_{\beta_1}^{m_1} \cdots f_{\beta_N}^{m_N} v_+ \}$ give a basis for $M(\lambda)$

Easy basis: $U(\mathfrak{g})$ is a free right $U(\mathfrak{b})$-module on basis

$$f_{\beta_1}^{m_1} \cdots f_{\beta_N}^{m_N}$$

where $\beta_1, \dots, \beta_N$ are positive roots

PBW theorem

f_β is "standard" basis vector for $\mathfrak{g}_{-\beta}$.

Then you show :-

$\{L(\lambda) \mid \lambda \in \mathbb{Z}^*\}$ gave all irred. h/w modules up to $\cong$.

- $M(\lambda)$ has a unique irreducible quotient $L(\lambda)$
- any irred h/w module of h/w λ is $\cong L(\lambda)$
- any f.d irred. $\mathfrak{g}$-module is obviously a h/w module

So the f.d. irred's are amongst the $L(\lambda)$'s. ✓

- $L(\lambda)$ is f.d. $\Longleftrightarrow$ λ is a <u>dominant weight</u>

HARD

If $L(\lambda)$ is f.d., consider $S_i = \langle e_i, h_i, f_i \rangle \cong sl_2(\mathbb{C})$

As $e_i v_+ = 0$ and $h_i v_+ = \lambda(h_i) v_+$, you deduce

that $\boxed{\lambda(h_i) = (\lambda, \alpha_i^\vee) \in \mathbb{N} \quad \forall i \in I}$

<u>Notation</u> Let $\omega_1, \omega_2, \dots, \omega_\ell$ be the dual basis

\varpi

to $\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_\ell^\vee$. Another basis for $E \subset \mathfrak{t}^*$.

Let $P = \{\lambda \in \mathfrak{t}^* \mid (\lambda, \alpha_i^\vee) \in \mathbb{Z} \ \forall i = 1, \dots, \ell\}$

<u>weight lattice</u>

$$= \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_\ell .$$

Let $P^+ = \{\lambda \in \mathfrak{t}^* \mid (\lambda, \alpha_i^\vee) \in \mathbb{N} \ \forall i = 1, \dots, \ell\}$

$$= \mathbb{N}\omega_1 \oplus \cdots \oplus \mathbb{N}\omega_\ell .$$

The ω_i's are <u>fundamental weights</u>.

Elements of P^+ are the <u>dominant weights</u>.

They label the h/w's of the f-d. irreps of $\mathfrak{g}$.

$\mathfrak{sl}_2(\mathbb{C})$ $\quad R = \{\pm\alpha\}$ $\quad \Delta = \{\alpha\}$

$E = \mathbb{R}\alpha^\vee$ $\quad (\alpha, \alpha^\vee) = 2$

Define $\omega \in E$ so $(\omega, \alpha^\vee) = 1$

$$\boxed{\omega = \frac{1}{2}\alpha}$$

Then $P^+ = \mathbb{N}\omega$

$L(n\omega)$ for $n \in \mathbb{N}$ ...

$$|||$$

$L(n)$ from before

$$h v_+ = n\omega(h) v_+ = (n\omega, \alpha^\vee) v_+ = n v_+$$

$\lambda \in P^+ \longleftrightarrow$ data $\lambda = \sum_{i=1}^{\ell} a_i \omega_i$ $\quad a_1, \dots, a_\ell \in \mathbb{N}$

$$a_i := (\lambda, \alpha_i^\vee) = \lambda(h_i)$$

$L(\lambda)$ f.d. irrep.

- Weyl character formula tells you $\dim L(\lambda)_\mu$ $\quad \lambda \in P^+$, $\mu \in P$

Call μ so $L(\lambda)_\mu \neq 0$ are of form $\lambda - \sum_{i=1}^{\ell} m_i \alpha_i$
with $m_i \in \mathbb{N}$ ... true even for $M(\lambda)$)

In particular, you see exactly which μ have $L(\lambda)_\mu \neq 0$.

Denote this by $Q \subseteq P(\lambda) \subseteq P$

root lattice
$\mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_\ell$

weight lattice
$\{\lambda \in \mathfrak{h}^* \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \ \forall i\}$
$\mathbb{Z}\omega_1 \oplus \dots \oplus \mathbb{Z}\omega_\ell$

- Weyl dimension formula

$$\dim L(\underset{P^+}{\underset{\cup}{\lambda}}) = \frac{\prod_{\alpha\in R^+}(\lambda+\rho, \alpha^\vee)}{\prod_{\alpha\in R^+}(\rho, \alpha^\vee)}$$

If $0\neq\lambda\in P^+$ and $\dim L(\lambda)$ is minimal, better have $\lambda=\omega_i$ for some i

$$\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha \iff (\rho, \alpha_i^\vee) = 1 \ \forall i \iff \rho = \omega_1+\omega_2+\dots+\omega_\ell.$$

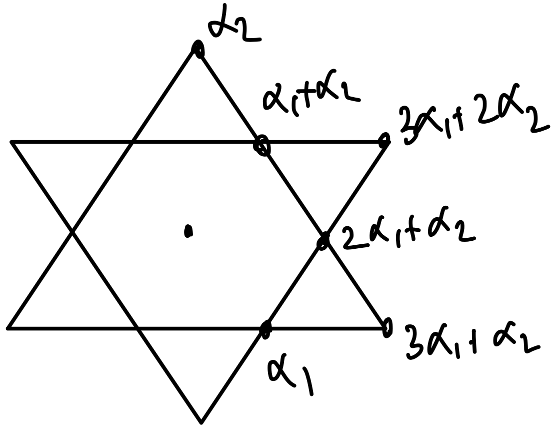
sum of fundamental wts.

eg G_2

adjoint representation $14 = 2+|R|$

$$\dim L(\omega_1) = \frac{1\cdot\cancel{3}\cdot\cancel{8}\cdot\cancel{5}\cdot 7\cdot 2}{1\cdot\cancel{2}\cdot\cancel{5}\cdot\cancel{3}\cdot\cancel{4}\cdot 1} = 14$$

$$\dim L(\omega_2) = \frac{\cancel{2}\cdot\cancel{3}\cdot 7\cdot\cancel{4}\cdot\cancel{5}\cdot 1}{1\cdot\cancel{2}\cdot\cancel{5}\cdot\cancel{3}\cdot\cancel{4}\cdot 1} = 7$$



$$\prod_{\alpha\in R^+}\frac{(\omega_1+2\omega_2, \alpha^\vee)}{(\omega_1+\omega_2, \alpha^\vee)}$$

minimal dimension from table in L9-2

- Tensor products $\lambda, \mu \in P^+$

$$L(\lambda) \otimes L(\mu) \cong \bigoplus_{\nu \in P} c_{\lambda\mu}^{\nu} \, L(\nu)$$

Multiplicities

My favorite way to calculate $c_{\lambda\mu}^{\nu}$'s goes via the theory of crystals (Kashiwara, 1991).

In type A_ℓ, $c_{\lambda\mu}^{\nu}$ = Littlewood-Richardson coefficients, counting tableaux.

Quantized enveloping algebras $U_q(\mathfrak{g})$