There are a few loose ends — I want to expand these in the last three lectures.

**PLAN**

1. A glimpse of representation theory

   So I can discuss Weyl dimension formula which is needed in proof of Lemma from L9-2.

2. More about exponential

   The cunning trick in the proof of Severi’s theorem is L9-3. Needs more discussion!

3. Construction of semisimple algebraic groups starting from semisimple Lie algebras.
A glimpse of representation theory

We developed $\text{Rep}(\mathfrak{sl}_2(\mathbb{C}))$ in detail already:

- Every f.d. $\mathfrak{sl}_2(\mathbb{C})$-module is Cor.
- Irreducible ones $/\sim$ are labelled by their highest weight $\lambda \in \mathbb{N}

$$L(n) = S^n V \quad \text{where } V = \mathbb{C}^2 \text{ is natural } 2\text{-d rep}.$$  

labels are $h$-eigenvalue of a basis weight

$$\begin{pmatrix} n \\ n-2 \\ n-4 \\ \vdots \\ 2-n \\ -n \end{pmatrix}$$

- $\dim L(n) = n+1$
- $h$ acts diagonally on any f.d. rep.
- $L(n) \otimes L(n) = \bigoplus_{p \leq \lfloor n/2 \rfloor} L(p)$,  
  $$\sum_{p \leq \lfloor n/2 \rfloor} L(p) \bigoplus_{p \equiv n+m \mod 2} L(p)$$

Clebsch-Gordan formula

HW 5-01
Let $\mathfrak{g}$ be a f.d. semisimple Lie algebra, \( \mathfrak{g} = \mathfrak{g}(\mathbb{C}) \) a Cartan matrix

\[ \Delta = \{ \alpha_1, \ldots, \alpha_d \} \subset RCE \text{ root system} \]

base

\[ \mathfrak{t} = \langle h_1, \ldots, h_d \rangle \text{ max. toral subalgebra} \]

\[ \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}^+ \]

\[ \langle e_i \rangle, \langle h_i \rangle, \langle e_i \rangle \text{ Cartan subalgebra} \]

\[ \langle e_i \rangle, \langle h_i \rangle, \langle e_i \rangle \text{ Cartan subalgebra} \]

\[ B = \mathfrak{t} \oplus \mathfrak{n}^+ \text{ Borel subalgebra } \]

corresponding to \( R^+ \subset \mathbb{R} \).

We know already:

- Any element in $\mathfrak{t}$ is semisimple, hence its spectral decomposition in any f.d. representation of $\mathfrak{g}$ (Jordan decomposition). As $\mathfrak{t}$ is Abelian, any $V \in \text{Rep}(\mathfrak{g})$ decomposes as $V = \bigoplus \nbigoplus \mathbb{C} \lambda$, for any $\lambda \in \mathbb{C}$. For all $\lambda, \lambda' \in \mathbb{C}$, $\lambda \neq \lambda'$, $\lambda' \neq 0$, $\langle e_i \rangle, \langle h_i \rangle, \langle e_i \rangle$ correspond to $\mathfrak{g}$.

- Any f.d. representation is c.r.
Def \( \lambda \in \mathbb{Z}^* \). A highest weight module of \( h/w \lambda \) is a \( \mathfrak{g}_3 \)-module generated by a non-zero highest weight vector \( v_+ \) of weight \( \lambda \).

There's a universal such module, the Verma module:

\[
M(\lambda) = U(\mathfrak{g}_3) \otimes U(\mathfrak{g}) \Lambda^{-\lambda}
\]

\( v_+ = (\otimes) h/w \) vector \( 1-D \) \( \mathfrak{g}_3 \)-module

\[
\{ f_{\beta_1}^{\mu_1} \ldots f_{\beta_n}^{\mu_n} v_+ \} \text{ give a basis for } M(\lambda)
\]

Easy basis: \( U(\mathfrak{g}_3) \) is a free right \( U(\mathfrak{g}) \)-module on basis

\[
f_{\beta_1}^{\mu_1} \ldots f_{\beta_n}^{\mu_n}
\]

PBW theorem: where \( \beta_1, \ldots, \beta_n \) are positive roots

\( f_{\beta} \) is "standard" basis vector for \( \mathfrak{g}_3 \).
Then you show:

- $M(\chi)$ has a unique module with quotient $L(\chi)$.
- Any $\text{rc} \, h/w$ module $W: \chi \rightarrow \chi \equiv L(\chi)$.
- Any $\text{f.d. } \text{rc} \, h/w$ module is obviously a $h/w$ module.
- So the $\text{f.d. } \text{rc} \, h/w$ modules are amongst the $L(\chi)$'s.

- $L(\chi)$ is f.d. $\iff \chi$ is a dominant weight.

If $L(\chi)$ is f.d., consider $S_\chi: = \left\langle e_i, h_i, f_i \right\rangle \cong \text{sl}_2(\mathbb{C})$.

As $e_i v_+ = 0$ and $h_i v_+ = \chi(h_i) v_+$, you deduce that $\chi(h_i) = (\chi, \delta_i^\vee) \in \mathbb{N} \quad \forall i \in I$. 

\[ \exists L(\chi) \mid \chi \in \mathbb{Z} \times \mathbb{Z} \text{ gain all } \text{rc} \, h/w \text{ module} \]
Notation: Let $\omega_1, \omega_2, \ldots, \omega_\ell$ be the dual basis.

Another basis for $\mathbb{Z}^\vee$ is $\lambda^i, \lambda^i, \ldots, \lambda^i$. Let $P = \{ \lambda \in \mathbb{Z}^\vee \mid (\lambda, \omega_i) \in \mathbb{Z} \forall i = 1, \ldots, \ell \}$

weight lattice

$\mathbb{Z} \omega_1 \oplus \cdots \oplus \mathbb{Z} \omega_\ell$.

Let $P^+ = \{ \lambda \in \mathbb{Z}^\vee \mid (\lambda, \omega_i) \in \mathbb{N} \forall i = 1, \ldots, \ell \}$

$\mathbb{N} \omega_1 \oplus \cdots \oplus \mathbb{N} \omega_\ell$.

The $\omega_i$'s are fundamental weights.

Elements of $P^+$ are the dominant weights.

They label the h/w's of the f.d. reps of $G$. 
\[ s \ell_2(\mathbb{C}) \quad R = \{ 3 \pm \alpha \} \quad \Delta = 3 \alpha \]
\[ E = 1 \mathbb{R} \alpha \quad (\alpha, \alpha^*) = 2 \]

Denote \( \omega \in E \) so \( (\omega, \alpha^*) = 1 \)
\[ \omega = \frac{1}{2} \alpha \]

Then \( p^+ = \mathbb{N} \omega \)

\[ L(n \omega) \text{ for } n \in \mathbb{N} \]
\[ L(n) \text{ from before} \]

\[ h v^+ = n \omega(h) v^+ \]
\[ = (n \omega, \alpha^*) v^+ \]
\[ = n v^+ \]
\lambda \in P^+ \rightarrow \text{ data } \chi = \sum_{\alpha_i} a_i \omega_i, \alpha_i \geq 0, \alpha_i \in \mathbb{N}

a_i = (\lambda, \alpha_i^\vee) = \lambda(h_i')

L(\lambda) \ f.d. \ irreps.

- Weyl character formula tells you that \( L(\chi) \mu \neq 0 \) \( \mu \in P \) is a form \( \lambda - \sum_{i=1}^{\ell} m_i \alpha_i \)

Call \( \mu \) so \( L(\chi)\mu \neq 0 \) are of form \( \lambda - \sum_{i=1}^{\ell} m_i \alpha_i \)

with \( m_i \in \mathbb{N} \) \( \mu \) is irreducible for \( M(\chi) \)

In particular, you see exactly which \( \mu \) have \( L(\chi)\mu \neq 0 \).

Denote this by \( Q \subseteq P(\chi) \subseteq P \)

\( \text{weight lattice} \ \{ \lambda \in \mathbb{Z}^\ast \mid (\lambda, \alpha_i^\vee) \in \mathbb{Z} \ \forall i \} \)

\( \text{root lattice} \ \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_\ell \)

\( \text{not lattice} \ \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_\ell \)
The Weyl dimension formula is:

\[ \dim L(\lambda) = \frac{\prod (\lambda + \rho, \alpha^+) \alpha \in \Phi^+}{\prod (\rho, \alpha^+)} \]

where \( \lambda \) is a dominant weight, \( \rho \) is the half-sum of positive roots, and \( \dim L(\lambda) \) is the dimension of the representation of \( \lambda \).

The character \( \rho \) is defined as:

\[ \rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \]

For the group \( G_2 \), the dimension is calculated as:

\[ \dim L(\omega_1) = \frac{\pi}{\omega_1 + 2\omega_2} \]
\[ \dim L(\omega_2) = \frac{\pi}{(\omega_1 + \omega_2)} \]

where \( \omega_1 \) and \( \omega_2 \) are fundamental weights.

The dimension of \( G_2 \) is 14, as calculated from the table in L9-2.
Tensor products \( \lambda, \mu \in \mathbb{P}^+ \)

\[
L(\lambda) \otimes L(\mu) = \bigoplus_{\nu \in \mathbb{P}} \nu^{\sim} L(\nu)
\]

Multiplicities

My favorite way to calculate \( \xi_{\lambda \mu} \)'s goes via the theory of crystals (Kashiwara, 1991).

In type \( A_n \), \( \xi_{\lambda \mu} = \) Littlewood-Richardson coefficients, coming from tableaux.

Quantized enveloping algebras \( U_q(\mathfrak{g}) \)