More about exponentiating

We saw this in proof of Serre's theorem. In fact its a foundational principle.

Let $x : V \to V$. Then $\exp(x) = 1 + x + \frac{x^2}{2!} + \dots : V \to V$

nilpotent linear map

f.d. vector space

Makes sense as actually finite sum

[If over $\mathbb{R}$ or $\mathbb{C}$ don't need nilpotency ... convergence]

Let $y : V \to V$ be another such.

Suppose x and y commute. Then

$$\boxed{\exp(x) \circ \exp(y) = \exp(x+y)}$$

follows from formal property of exponential power series

algebra: binomial theorem ex.

analysis: laws of exponential

This implies $\exp(x) \in \cancel{GL}(V)$ SL

as $\exp(x)^{-1} = \exp(-x)$

If a unipotent linear map — all its eigenvalues are 1

$$\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

- If V is an algebra A and $x : A \to A$ is a derivation then $\exp(x) : A \to A$ is an algebra automorphism.

$$\Big(1 + x + \frac{x^2}{2!} + \cdots\Big)(a\,b) = ab + x(a)b + a\,x(b) + \frac{x^2}{2!}(a)\,b + x(a)x(b) + a\frac{x^2}{2!}(b) + \cdots$$

$$= \Big(a + x(a) + \frac{x^2}{2!}(a) + \cdots\Big)\Big(b + x(b) + \frac{x^2}{2!}(b) + \cdots\Big) = \cdots$$

$\Rightarrow$ $ad\,x$ is nilpotent

- If $x, y \in \mathfrak{g} < \mathfrak{gl}(V)$, V f.d., x nilpotent

matrix mult.

$GL(\mathfrak{g})$

$$\boxed{\exp(x)\, y\, \exp(x)^{-1} = \exp(ad\, x)(y)} \Longrightarrow Ad(\exp x) = \exp(ad\, x) \text{ in } \mathrm{Inn}(\mathfrak{g}) \leq \mathrm{Aut}(\mathfrak{g})$$

$Ad(\exp x)(y)$

Lie alg-autos.

$ad\,x = \lambda_x - \rho_x$ left/right mult. by x, commute

$\therefore \exp(ad\,x) = \exp(\lambda_x) \circ \exp(\rho_x)^{-1}$

<u>Example</u>) $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ e, h, f as usual.

Let exponentiate te, tf in natural representation, any $t \in \mathbb{C}$.

$$\exp(te) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}}_{x(t) \in SL_2(\mathbb{C})}$$

$$\exp(tf) = \underbrace{\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}}_{y(t) \in SL_2(\mathbb{C})}$$

$x(t)x(t') = x(t+t')$

So: $x : \mathbb{G}_a \to SL_2(\mathbb{C})$

$y : \mathbb{G}_a \to SL_2(\mathbb{C})$ ← a <u>morphism of alg. gps.</u>

Images of x & y: "1-parameter subgroups" or "root groups"

Now observe $SL_2(\mathbb{C}) = \langle \underbrace{x(t)}_{X}, \underbrace{y(t)}_{Y} \mid t \in \mathbb{C} \rangle$.

$L(X) = \langle e \rangle$ $L(Y) = \langle f \rangle$

HW1-Q2 : Given a family $x_i : \mathbb{G}_a \to GL_n(\mathbb{C})$ $(i \in I)$ of morphisms of algebraic groups, the subgroup generated by their images is closed & connected, hence, a connected alg. gp. itself.

Take $t \in \mathbb{C}^\times$. Let $\boxed{\sigma(t) = x(t)\, y(-t^{-1})\, x(t)}$

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & t \\ -t^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}}_{\sigma(t) \in SL_2(\mathbb{C})}$$

I'll write $\boxed{\sigma = \sigma(1)}$ $\qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{C})$

Define $\boxed{\gamma(t) = \sigma(t)\,\sigma(-1)}$

$$\begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in SL_2(\mathbb{C})$$

$\gamma : \mathbb{G}_m \to SL_2(\mathbb{C})$ another morphism of algebraic groups

Given a 1-psg $T = \{\gamma(t) \mid t \in \mathbb{C}^\times\}$ 1-D torus $L(T) = \mathbb{Z} = \langle h \rangle$

Example 2 $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ in representation $\mathrm{ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$

$$\exp(\mathrm{ad}\ te) = \begin{pmatrix} 1 & -2t & -t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \in PSL_2(\mathbb{C}) \qquad \mathrm{ad}\, te = \begin{array}{c} \begin{array}{ccc} e & h & f \end{array} \\ \begin{pmatrix} 0 & -2t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

$\mathrm{Ad}(x(t))$

$$\exp(\mathrm{ad}\ tf) = \begin{pmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ -t^2 & 2t & 1 \end{pmatrix} \in PSL_2(\mathbb{C}) \qquad \mathrm{ad}\, tf = \begin{pmatrix} 0 & 0 & 0 \\ -t & 0 & 0 \\ 0 & 2t & 0 \end{pmatrix}$$

$\mathrm{Ad}(y(t))$

$$\mathrm{Ad}(\sigma(t)) = \begin{pmatrix} 0 & 0 & -t^2 \\ 0 & -1 & 0 \\ -t^{-2} & 0 & 0 \end{pmatrix} \in PSL_2(\mathbb{C}) \qquad \mathrm{Ad}(\sigma) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$h \mapsto -h$

$$\mathrm{Ad}(\tau(t)) = \begin{pmatrix} t^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-2} \end{pmatrix} \in PSL_2(\mathbb{C})$$

$$\mathrm{Ad}(\underset{\substack{\uparrow \\ \mathrm{Ad}\, T}}{\tau}(t))(e) = t^2 e \qquad (\mathrm{ad}\, \underset{\substack{\uparrow \\ \mathrm{ad}\, \mathfrak{t}}}{h})(e) = 2e$$

Example 3 Cartan matrix $C = ((\alpha_i, \alpha_j^\vee))_{i,j=1,\dots,\ell}$

$\mathfrak{g} = \mathfrak{g}(C)$ as given by Serre presentation.

$S_i = \langle e_i, h_i, f_i \rangle \qquad \sigma_i = (\exp \operatorname{ad} e_i)(\exp \operatorname{ad}(-f_i))(\exp \operatorname{ad} e_i) \in \operatorname{Aut}(\mathfrak{g})$

On h_i, $\sigma_i(h_i) = -h_i$

On $\ker(\alpha_i) < \mathfrak{t} = \langle h_1, \dots, h_\ell \rangle$, σ_i is the identity
(codim-1 subspace)

$\Rightarrow \sigma_i$ acts on $\mathfrak{t}$ sending $h_i \mapsto -h_i$, fixing $\ker(\alpha_i)$

$\Rightarrow \lambda(\underset{\overset{\uparrow}{\mathfrak{t}}}{\sigma_i^{-1}(h)}) = \underbrace{s_i(\lambda)}_{\lambda - (\lambda, \alpha_i^\vee)\alpha_i}(h) \quad \forall \lambda \in \mathfrak{t}^*$

step missing in proof of Serre in L9-3

$[h, \sigma_i(x)] = \sigma_i[\sigma_i^{-1}(h), x] = \lambda(\sigma_i^{-1}(h))\,\sigma_i(x)$
$= (s_i(\lambda))(h)\,(\sigma_i(x))$

$\Rightarrow \sigma_i(\underset{\overset{\cup}{x}}{\mathfrak{g}_\lambda}) = \mathfrak{g}_{s_i(\lambda)}$

<u>Example 4</u> Still $\mathfrak{g} = \mathfrak{g}(C)$ f.d. semisimple Lie algebra.

Let V any f.d. $\mathfrak{g}$-module. $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$

Let $\sigma_i = \exp \rho(e_i) \exp \rho(-f_i) \exp \rho(e_i)$

Arguments above show ... $\sigma_i(V_\lambda) = V_{s_i(\lambda)}$

s_i: ith simple reflection in W

$$\implies \dim V_\lambda = \dim V_{w(\lambda)} \quad \forall w \in W$$

Weight space dimensions are constant on W-orbits