Outline: classification of semisimple algebraic groups over $\mathbb{k} = \overline{\mathbb{k}}$, any characteristic.

connected alg. group G with no connected closed normal solvable subgroups

Over $\mathbb{k} = \mathbb{C}$, G is semisimple $\iff \mathfrak{g}$ is semisimple, so there's a root system $R \subset E$ in the background.

The classification of G need more data than that, eg $SL_2(\mathbb{C})$, $PSL_2(\mathbb{C})$ $\to \mathfrak{sl}_2(\mathbb{C})$

Additional data!!

Let $R \subset E$ be a root system, $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ a base.

$$Q \leq P$$

root lattice $= \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i$, weight lattice $= \bigoplus_{i=1}^{\ell} \mathbb{Z}\omega_i$, fundamental weights

- $(\omega_i, \alpha_j^\vee) = \delta_{ij}$
- $\alpha_i = \sum_{j=1}^{\ell} c_{ij}\,\omega_j$

$c_{ij} = (\alpha_i, \alpha_j^\vee)$ Cartan integer

$C = (c_{ij})_{1 \leq i,j \leq \ell}$ Cartan matrix

$\Longrightarrow$ P/Q finite Abelian group of order $|P/Q| = \det C$

Could determine exactly by finding elementary divisors of C...

This group P/Q is <u>the fundamental group</u> of the root system

An isomorphism $f: (R \subset E) \xrightarrow{\sim} (R' \subset E')$ of root systems takes Q to Q', P to P', so induces an $\cong$ of fundamental groups.

<u>Theorem</u> There's a bijection

$$\left\{ \begin{array}{c} \text{semisimple algebraic groups} \\ \text{over } \Bbbk = \bar{\Bbbk} \end{array} \right\} \Big/ \cong \quad \xrightarrow{\sim} \quad \left\{ \text{pairs } ((R, E), \Gamma) \right\} \Big/ \cong$$

root system

subgroup of its fundamental group P/Q

Examples

① $\mathfrak{sl}_4(\mathbb{C})$ A_3

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$\det C = 4$

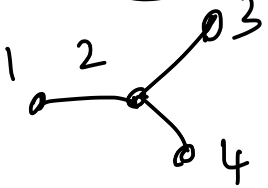
$$\begin{tikzcd}[row sep=small] P/Q \cong C_4 \arrow[d, dash] \\ C_2 \arrow[d, dash] \\ 1 \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] SL_4(\mathbb{C}) \arrow[d, dash] \\ SL_4(\mathbb{C})/\langle \pm I \rangle \arrow[d, dash] \\ PSL_4(\mathbb{C}) \end{tikzcd}$$

$\leftarrow \Gamma = P/Q$ simply connected

$Z(SL_4(\mathbb{C})) \cong C_4$

② $\mathfrak{so}_8(\mathbb{C})$ D_4



$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$$

$\det C = 4$

$$\begin{tikzcd}[column sep=small] & P/Q \cong C_2 \times C_2 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ C_2 \arrow[dr, dash] & C_2 \arrow[d, dash] & C_2 \arrow[dl, dash] \\ & 1 & \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] Spin_8(\mathbb{C}) \arrow[d, dash] \\ SO_8(\mathbb{C}) \arrow[d, dash] \\ PSO_8(\mathbb{C}) \end{tikzcd}$$

$Out(Spin_8(\mathbb{C})) \cong S_3$

symmetries of Dynkin diagram

τ, $\tau^3 = 1$ triality automorphism of $Spin_8(\mathbb{C})$

The forward map is this classification theorem.

Take G, semisimple algebraic group.

Pick T, a maximal torus, $T \cong \underbrace{\mathbb{G}_m \times \cdots \times \mathbb{G}_m}_{l = \text{rank} = \dim T}$

Let $\mathfrak{g} = L(G)$, $\mathfrak{t} = L(T)$. Consider adjoint action of T on $\mathfrak{g}$.

Get decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

If over $\mathbb{C}$, $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{g}$, $X(T) \hookrightarrow \mathfrak{t}^*$, this is Cartan decomp.!

where for $\lambda \in X(T) = \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^l$ free Abelian group

character group of T

we write $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid (\mathrm{Ad}\, t)(x) = \lambda(t)x \;\; \forall\, t \in T\}$

and $R = \{0 \neq \alpha \in X(T) \mid \mathfrak{g}_\alpha \neq 0\}$

So now we have $R \subset X(T)$, free Abelian group

$X(T) \subset E \subset \mathfrak{t}^*$

in Lie algebras, had $R \subset \mathfrak{t}^*$, complex vector space

Now $R \subset E$, where $E = \mathbb{R} \otimes_{\mathbb{Z}} X(T)$.

Want to make E into a Euclidean space so that this is a root system.

Define $W = N_G(T)/T$ — this is a finite group! It acts naturally on $X(T)$, hence, on E by $\mathbb{R}$-linear autos.

Pick a W-invariant inner product on E.

Show that $(R \subset E)$ is abstract root system, and W is its Weyl group.

Finally we need $\Gamma \leq P/Q$, subgroup of fundamental group of this root system. That's just $X(T)/Q$

$Q \leq X(T) \leq P$

This then is the map in the theorem!

<u>What about the other direction</u>? Let's start over $\mathbb{C}$.

Take $R \subset E$, $\Gamma \leq P/Q$, want to construct corresponding group G.

- Pick base $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$, hence, Cartan matrix C, $\mathfrak{g} = \mathfrak{g}(C)$, triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.
- Pick $\lambda \in P^+$ so $P(\lambda)/Q = \Gamma$, let $V = L(\lambda)$ for short.

$P(\lambda)$ = set of weights of f.d. irreducible $L(\lambda)$ of h/w λ

$Q \leq P(\lambda) \subseteq P$

- Then exponentiate $\mathfrak{g}$ in the representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.

Set $x_i(t) = \exp(\rho(te_i))$, $y_i(t) = \exp(\rho(tf_i))$

Then $G = \langle x_i(t), y_i(t) \mid t \in \mathbb{C}, i = 1, \dots, \ell \rangle < GL(V)$

connected algebraic group / $\mathbb{C}$

It turns out this is semisimple alg. group realizing $(R < E)$, Π.

Lots of work is needed to prove all this!!

What about other fields?

In general, need Chevalley construction of Chevalley groups

some signs to choose carefully.

Pick Chevalley basis for $\mathfrak{g}$.

$$\{e_\alpha \mid \alpha \in R\} \cup \{h_1, \dots, h_\ell\} \qquad h_i = h_{\alpha_i}$$

let $f_\alpha = e_{-\alpha}$ for short, $h_\alpha := [e_\alpha, f_\alpha]$

then $\langle e_\alpha, h_\alpha, f_\alpha \rangle$'s are sl_2-triples used before

You can pick this so all Lie algebra structure constants are in $\mathbb{Z}$.

$$[e_\alpha, e_\beta] = \begin{cases} \pm(r+1)\, e_{\alpha+\beta} & \text{if } \alpha+\beta \in R, \\ & \beta - r\alpha, \dots, \beta + q\alpha \quad \alpha\text{-string through } \beta \\ 0 & \text{if } \alpha+\beta \notin R \end{cases}$$

α, β linearly independent

Assume $\Gamma = P/Q$, ie just construct simply connected G. (universal Chevalley group)

Let $x_\alpha(t) = \exp(\rho(te_\alpha))$ from Chevalley basis

Then $G = \langle x_\alpha(t) \mid \alpha \in R, t \in \mathbb{C} \rangle$ and the following relations hold:

① $x_\alpha(t) x_\alpha(t') = x_\alpha(t+t')$ $\quad \alpha \in R, \; t, t' \in \mathbb{C}$

Chevalley commutator formula

② $(x_\alpha(t), x_\beta(t')) = \overrightarrow{\prod_{\substack{i,j \geq 1 \\ i\alpha + j\beta \in R}}} x_{i\alpha+j\beta}(c_{ij} t^i (t')^j)$

constants depending on i,j, α, β, order of $\overrightarrow{\prod}$, signs in Chevalley basis

$c_{ij} \in \mathbb{Z}$

α, β linearly independent $\quad (g,h) = ghg^{-1}h^{-1}$ group commutator

(3) $\sigma_\alpha(t)\, x_\alpha(t')\, \sigma_\alpha(t)^{-1} = x_{-\alpha}(-t^2 t')$

$t \in \mathbb{C}^\times$, $t' \in \mathbb{C}$

where $\sigma_\alpha(t) = x_\alpha(t)\, x_{-\alpha}(-t^{-1})\, x_\alpha(t) \in G$

(4) $\gamma_\alpha(t)\, \gamma_\alpha(t') = \gamma_\alpha(t t')$

$t, t' \in \mathbb{C}^\times$

where $\gamma_\alpha(t) = \sigma_\alpha(t)\, \sigma_\alpha(-1)$

Get group $G(\mathbb{k})$ any field $\mathbb{k}$

<u>Universal Chevalley group over $\mathbb{k}$</u>

$G(q)$ (almost) finite simple groups

<u>Miracle</u>: These give a complete set of relations for group G.

The relations make sense with $\mathbb{C}$ replaced by $\mathbb{k}$

(indeed, <u>any</u> field, not even algebraically closed)

<u>finite groups of Lie type</u>