Examples of algebraic groups

We've already seen $G = GL_n(\mathbb{k})$

For this,
- $\mathbb{k}[G] = \mathbb{k}[T_{ij} \mid 1 \leq i,j \leq n]_{det}$
- $m^* : \mathbb{k}[G] \to \mathbb{k}[G] \otimes \mathbb{k}[G]$, $\quad T_{ij} \mapsto \sum_{k=1}^{n} T_{ik} \otimes T_{kj}$
- $ev_e : \mathbb{k}[G] \to \mathbb{k}$, $\quad T_{ij} \mapsto \delta_{ij}$
- Antipode $\quad i^*(T_{ij})$ is that nasty formula for ij-entry of inverse of an $n \times n$ matrix.

Any finite group is an algebraic group

Any closed subgroup of $GL_n(\mathbb{k})$ is again an algebraic.

- $SL_n(\mathbb{k}) = \{g \in GL_n(\mathbb{k}) \mid \det g = 1\}$ $\quad I = \langle \det - 1 \rangle \in \mathbb{k}[GL_n(\mathbb{k})]$
- $B_n(\mathbb{k}) = \{g \in GL_n(\mathbb{k}) \mid g_{ij} = 0 \text{ for } i > j\}$ $\quad I = \langle T_{ij} \mid i > j \rangle$

  "upper triangular invertible matrices"
- $T_n(\mathbb{k}) : \{g \in GL_n(\mathbb{k}) \mid g_{ij} = 0 \text{ for } i \neq j\}$ $\quad I = \langle T_{ij} \mid i \neq j \rangle$

  "diagonal invertible matrices"

- $U_n(\mathbb{k}) := \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \triangleleft B_n(\mathbb{k})$, $B_n(\mathbb{k}) / U_n(\mathbb{k}) \cong T_n(\mathbb{k})$

"uni-triangular"

$$I = \langle T_{ij}, T_{kk} - 1 \mid i > j, \ k = 1, \dots, n \rangle$$

Dimensions?

$GL_n(\mathbb{k})$ dim n^2

$SL_n(\mathbb{k})$ dim $n^2 - 1$

$B_n(\mathbb{k})$ dim $\frac{1}{2}n(n+1)$

$T_n(\mathbb{k})$ dim n

$U_n(\mathbb{k})$ dim $\frac{1}{2}n(n-1)$

Any centralizer or any normalizer of any closed subset of alg. group G is a closed subgroup.

Classical groups. Let $(\cdot,\cdot)$ be a non-degenerate symmetric or skew-symmetric bilinear form on $\mathbb{k}^n$.

Implies $n = 2m$, even.

Let $J = (J_{ij})_{1 \le i,j \le n}$ be Gram matrix

so $J_{ij} = (v_i, v_j)$ where $v_1, \dots, v_n$ standard basis

Then
$$G = \{ g \in GL_n(\mathbb{k}) \mid (gv, gw) = (v,w) \ \forall v,w \in V \}$$
$$= \{ g \in GL_n(\mathbb{k}) \mid g^T J g = J \}$$

(a closed subgroup, hence, an alg. group).

Symmetric case: G is orthogonal group

$$J = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = I_n$$
$$g^T g = I_n$$

$$J = \begin{pmatrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{pmatrix}$$
$$g^T J g = J$$

Better choice for our canonical Gram matrix in orthogonal case

Call this $O_n(\mathbb{k})$

Skew-symmetric case: G is symplectic group

$$J = \begin{pmatrix} \begin{smallmatrix}0&1\\-1&0\end{smallmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{smallmatrix}0&1\\-1&0\end{smallmatrix} \end{pmatrix}$$

$$J = \begin{array}{c} \begin{array}{cc} m \qquad & \qquad m \end{array} \\ \left(\begin{array}{c|c} 0 & \begin{smallmatrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} 0 & & -1 \\ & ⋰ & \\ -1 & & 0 \end{smallmatrix} & 0 \end{array}\right) \end{array}$$

I like this one!

Call G: $Sp_{2m}(k)$

There's some stuff about $O_n(k)$, $Sp_{2m}(k)$ on HW 1.

$g \in O_n(k)$
$\det(g) = \pm 1$

$g \in Sp_{2m}(k)$
$\det(g) = 1$

$$1 \to \underbrace{SO_n(k)}_{\text{ker det, special orthogonal group.}} \to O_n(k) \xrightarrow{\det} \{\pm 1\} \to 1$$

$\left. \begin{array}{l} Sp_{2m}(k) \\ SO_n(k) \end{array} \right\}$ both connected/irreducible varieties.

Dimensions?

Cayley's theorem for algebraic groups...

Theorem Every algebraic group G is linear, i.e., it is isomorphic to a closed subgroup of $GL_n(\Bbbk)$ for some n.

careful!

Proof We need a f.d. vector space V on which G acts. Look in $\Bbbk[G]$!!!

For $g \in G$, have
$$\rho_g : \Bbbk[G] \to \Bbbk[G]$$
$$f \mapsto \rho_g f$$
$$(\rho_g f)(h) = f(hg)$$

NOT an algebraic group

$\Rightarrow \rho : G \to GL(\Bbbk[G])$ group homomorphism

right regular representation of G.

Claim Any f.d. $W \leq \Bbbk[G]$ lies in a f.d. subspace $V \leq \Bbbk[G]$ which is invariant under ρ_g for all $g \in G$.

Lemma X affine variety. $a_1, \dots, a_m \in \Bbbk[X]$ linearly independent

$\exists\, x_1, \dots, x_m \in X$ such that
$$\left(a_i(x_j)\right)_{1 \leq i,j \leq m}$$
is invertible matrix.

Pf of Claim. It suffices to prove the case $W = \langle f \rangle$.

Let $V = \langle \rho_g f \mid g \in G \rangle$ — invariant.

Need to show $\dim V < \infty$.

Let $m^*(f) = \sum_{i=1}^{n} f_i \otimes f_i'$, some n and $f_i, f_i' \in \mathbb{k}[G]$.

$$(\rho_g f)(h) = f(hg) = \sum_{i=1}^{n} f_i \otimes f_i'((h,g)) = \sum_{i=1}^{n} f_i(h) f_i'(g)$$

$$\Rightarrow \rho_g f = \sum_{i=1}^{n} f_i'(g) f_i \in \langle f_1, \dots, f_n \rangle$$

$$\Rightarrow V \leq \langle f_1, \dots, f_n \rangle \Rightarrow \dim V \leq n \quad \checkmark$$

Now pick $f_1, \dots, f_n \in \mathbb{k}[G]$, linearly independent algebra generators.

Apply claim to $W = \langle f_1, \dots, f_n \rangle$, WMA that $V = \langle f_1, \dots, f_n \rangle$

is invariant under all ρ_g's, and $f_1, \dots, f_n$ lin. ind. algebra generators for $\mathbb{k}[G]$.

$$G \longrightarrow GL(V) \qquad \text{group homomorphism.}$$
$$g \longmapsto \rho_g|_V$$

Claim $m^*(V) \subseteq V \otimes \Bbbk[G]$

b_i's lin. ind.

Pf Take $f \in V$. Say $m^*(f) = \underbrace{\sum_{i=1}^m a_i \otimes b_i}_{\in V \otimes \Bbbk[G]}$, $a_i, b_i \in \Bbbk[G]$.

As in previous claim, $\rho_g f = \sum_{i=1}^m b_i(g) a_i$.

Apply Lemma to get $g_1, \dots, g_m \in G$ so $(b_i(g_j))_{i,j=1,\dots,m}$ is invertible.

A lin. comb. of $\underset{\overset{\cap}{V}}{\rho_{g_j} f} = \sum_{i=1}^m b_i(g_j) a_i$ gives $a_i \in V \quad \forall\, i=1,\dots,m$.

This proves claim //

By claim, $m^*(f_j) = \sum_{i=1}^{\hat{n}} f_i \otimes f_{ij}$, some $f_{ij} \in \mathbb{k}[G]$.

↑ evaluate at (e, g) to get $f_j(g) = \sum_{i=1}^{n} f_i(e) f_{ij}(g) \Rightarrow f_j \overset{(*)}{=} \sum f_i(e) f_{ij}$

$V = \langle f_1, \dots, f_n \rangle$

$$\varphi : G \longrightarrow GL(V) \cong GL_n(\mathbb{k})$$
$$g \mapsto \rho_g|_V = (f_{ij}(g))_{i,j=1,\dots,n}$$

group homomorphism

morphism of affine varieties

This shows that $\varphi^* : \mathbb{k}[GL_n(\mathbb{k})] \longrightarrow \mathbb{k}[G]$

takes $T_{ij} \longmapsto f_{ij}$

i.e. that φ is a morphism of varieties.

φ is closed immersion, so iso. between G and its image, closed subgroup of $GL_n(\mathbb{k})$

In fact, φ^* is surjective as f_{ij}'s generate $\mathbb{k}[G]$, which follows as $f_j \in \langle f_{ij} \rangle$ by $(*)$, and f_j's generate $\mathbb{k}[G]$ by original choice.