Tangent space

Let X be an affine variety, $x \in X$ $\rightsquigarrow$ $T_x(X)$ tangent space to X at x.

Assume to start with given coordinates

$$x \in X \underset{\substack{\text{closed}\\ \text{embedding}}}{\subseteq} \mathbb{A}^n \longleftrightarrow \mathbb{k}[T_1, \dots, T_n]/I = \mathbb{k}[X]$$

$$I = I(X) = \langle f_1, \dots, f_s \rangle$$

Take a line L in $\mathbb{A}^n$ passing through x
$\quad \| \; x + tv$ for $t \in \mathbb{k}$ and $v = (v_1, \dots, v_n) \in \mathbb{k}^n$

Then $L \cap X$ is all $x + tv$ such that $f_i(x+tv) = 0 \quad (i = 1, \dots, s)$

$t = 0$ is a solution

Call L a <u>tangent line</u> to X at x if $t = 0$ is a multiple root of these equations.

Call v a <u>tangent vector</u>

Then $T_x(X) = \{ \text{all tangent vectors } v \in \mathbb{k}^n \}$

$$\frac{\partial}{\partial T_i} : \mathbb{k}[T_1,\dots,T_n] \longrightarrow \mathbb{k}[T_1,\dots,T_n]$$

Taylor expansion of $f_i : \mathbb{k}^n \to \mathbb{k}$ at x:

$$f_i(x+tv) = t \sum_{j=1}^{n} \frac{\partial f_i}{\partial T_j}(x)\, v_j + t^2 \boxed{} + \cdots$$

Saying "$t=0$ is a multiple root" means the linear term is zero for all i.

So: $\quad v \in T_x(X) \iff \boxed{\sum_{j=1}^{n} \frac{\partial f_i}{\partial T_j}(x)\, v_j = 0} \quad (i=1,\dots,s)$

provisional definition!

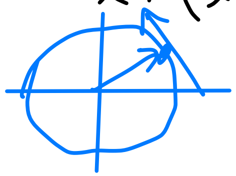
$$T_x(X) = \ker\left(\frac{\partial f_i}{\partial T_j}(x)\right)_{\substack{i=1,\dots,s\\ j=1,\dots,n}} \leq \mathbb{k}^n \qquad s\times n \text{ matrix}$$

eg (1) $X=\mathbb{A}^n$, $s=0$

$T_x(X) = \mathbb{k}^n$

(2) $X = V(T_1^2+T_2^2-1) \subseteq \mathbb{A}^2$ $\quad \ker(2x_1 \;\; 2x_2)$

$x = (x_1,\dots,x_n) \in X$ $\quad \| \quad \langle (x_2,-x_1) \rangle$



$p=2$? $\quad T_1^2+T_2^2-1 = (T_1+T_2+1)^2$ wrong f!!

Def $D : \mathbb{k}[X] \to \mathbb{k}[X]$ linear map is a derivation if

Leibniz holds: $D(fg) = D(f)g + f D(g)$ $\leftarrow$ $D(1) = 0$

Let $\mathrm{Der}(\mathbb{k}[X], \mathbb{k}[X])$ be vector space of all derivations

$\leq \mathrm{End}_{\mathbb{k}}(\mathbb{k}[X])$

Not an associative subalgebra

It is a Lie subalgebra!!!

(Commutator $[D_1, D_2]$ of two derivations is a derivation.)

assoc. algebra ..., so a Lie algebra via $[\cdot,\cdot]$

then its called $\mathfrak{gl}(\mathbb{k}[X])$

$\Downarrow$ If $D : \mathbb{k}[X] \to \mathbb{k}[X]$ is a derivation then $\boxed{\overline{D} = \mathrm{ev}_x \circ D}$ is a point derivation

For $x \in X$, let $\mathbb{k}_x = \mathbb{k}[X] / \mathfrak{m}_x$ $\leftarrow$ $\mathfrak{m}_x = \ker \mathrm{ev}_x$

Let $\mathrm{Der}(\mathbb{k}[X], \mathbb{k}_x) = \{ \overline{D} : \mathbb{k}[X] \to \mathbb{k}_x \mid \overline{D}(fg) = \overline{D}(f)g(x) + f(x)\overline{D}(g) \}$

vector space of point derivations for point x

$\mathrm{Der}(\mathbb{k}[T_1,\dots,T_n], \mathbb{k}_x)$?

$$\mathbb{k}^n \xrightarrow[(\dagger)]{\sim} \mathrm{Der}(\mathbb{k}[T_1,\dots,T_n], \mathbb{k}_x)$$

$v = (v_1,\dots,v_n) \in \mathbb{k}^n$

$$v \longmapsto \overline{D}_v$$

$$\boxed{ev_x \circ \sum_{j=1}^{n} v_j \frac{\partial}{\partial T_j}}$$

call this $\overline{D}_v$

This is an isomorphism

Proof: Any $\overline{D} \in \mathrm{Der}(\mathbb{k}[T_1,\dots,T_n], \mathbb{k}_x)$ equals $\overline{D}_v$ where $v_j := \overline{D}(T_j)$.

Now see $\overline{D} \longmapsto v$ for v defined in this way is 2-sided inverse //

Earlier we showed $T_x(X) = \{v \in \mathbb{k}^n \mid \overline{D}_v f_i = 0 \quad i = 1,\dots,s\}$

$$= \{v \in \mathbb{k}^n \mid \overline{D}_v I(X) = 0\}$$

$$\cong \downarrow (\dagger)$$

Our $T_x(X)$ (up to iso ...)

$$\boxed{\mathrm{Der}(\mathbb{k}[X], \mathbb{k}_x)} \cong \{\overline{D} \in \mathrm{Der}(\mathbb{k}[T_1,\dots,T_n], \mathbb{k}_x) \mid \overline{D}\, I(X) = 0\}$$

<u>Def</u>: X affine, $x \in X$. The <u>tangent space</u> to X at x is the vector space

$$T_x(X) := \mathrm{Der}(\mathbb{k}[X], \mathbb{k}_x)$$

(point derivations for the point x)

Coordinate free !! Functorial ...

If $\varphi : X \to Y$ morphism of affine varieties, get <u>linear map</u>

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \cup\!\!\!\!\!| & & \cup\!\!\!\!\!| \\ x & \mapsto & y \end{array}$$

$$d\varphi_x : T_x(X) \to T_x(Y) \qquad \text{\underline{differential} of } \varphi \text{ at } x$$

$$\overline{D} \mapsto (d\varphi_x)(\overline{D}) \quad \text{where} \quad (d\varphi_x)(\overline{D})(\underset{\mathbb{k}[Y]}{\underset{\cap}{f}}) := \overline{D}(\underset{\mathbb{k}[X]}{\underset{\cap}{\varphi^* f}})$$

Why is this a point derivation $\mathbb{k}[Y] \to \mathbb{k}_y$?

$$\overline{D}(\varphi^*(fg)) = \overline{D}(\varphi^* f \cdot \varphi^* g) = \overline{D}(\varphi^* f)(\varphi^* g)(x) + (\varphi^* f)(x)\,\overline{D}(\varphi^* g)$$

$$= \overline{D}(\varphi^* f)\, g(y) + f(y)\, \overline{D}(\varphi^* g) \quad \checkmark$$

If
$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & Y & \xrightarrow{\theta} & Z \\ \cup & & \cup & & \cup \\ x & \longmapsto & y & \longmapsto & z \end{array}$$

$$d(\theta\circ\varphi)_x : T_x(X) \to T_z(Z)$$
$$\|$$
$$d\theta_y \circ d\varphi_x$$

"chain rule"

(as $(\theta\circ\varphi)^* = \varphi^*\circ\theta^*$).

Passing to tangent space / differential is a functor from pointed aff. varieties to fin.d. vector spaces.

We just saw exactly this for $X \hookrightarrow \mathbb{A}^n$

Other observations

- $T_{(x,y)}(X\times Y) = T_x(X) \oplus T_y(Y)$
- If $\varphi: Z \hookrightarrow X$ is a closed embedding then $d\varphi_z : T_z(Z) \to T_z(X)$ is isomorphic $T_z(Z) \cong \{ D \in T_z(X) \mid D\, I(Z) = 0 \}$

- $f \in \Bbbk[X]$, $x \in D(f)$ i.e. $f(x) \neq 0$

$i : D(f) \hookrightarrow X$. Its differential is an isomorphism

$$di_x : T_x(D(f)) \xrightarrow{\sim} T_x(X)$$

"tangent space only sees a neighborhood of x"

$$\Bbbk[D(f)] = \Bbbk[X]_f$$

Have $\operatorname{Der}(\Bbbk[X]_f, \Bbbk_x) \longrightarrow \operatorname{Der}(\Bbbk[X], \Bbbk_x)$

"restriction along $i^* : \Bbbk[X] \to \Bbbk[X]_f$"

We need to show any $\overline{D} \in \operatorname{Der}(\Bbbk[X], \Bbbk_x)$ "lifts" to $\Bbbk[X]_f \to \Bbbk_x$

Issue is to define

$$\overline{D}\left(\frac{g}{f}\right) = \frac{\overline{D}(g) f(x) - g(x) \overline{D}(f)}{f(x)^2} \quad \ldots$$

$\overline{D}\left(\frac{g}{f} f\right)$ ...no choice

makes sense as $f(x) \neq 0$

eg $T_e(GL_n(\mathbb{k})) = T_e(M_n(\mathbb{k})) \cong M_n(\mathbb{k})$

$ev_e \circ \frac{\partial}{\partial T_{ij}} \in Der(\mathbb{k}[M_n(\mathbb{k})], \mathbb{k}_e) \longleftrightarrow e_{ij} \in M_n(\mathbb{k})$ ij-matrix

linear basis

$\mathbb{k}[M_n(\mathbb{k})] = \mathbb{k}[T_{ij} \mid 1 \leq i,j \leq n]$

eg $T_e(SL_n(\mathbb{k})) = \{ \cancel{v} \in M_n(\mathbb{k}) \mid D_{\cancel{v}\,A}(\det - 1) = 0 \} = \{A \in M_n(\mathbb{k}) \mid trA = 0\}$

$V(\det - 1)$

$A = (a_{ij})_{1 \leq i,j \leq n} = \sum a_{ij} e_{ij}$

all matrices $A \in M_n(\mathbb{k})$ s.t. $\sum_{i,j=1}^{n} a_{ij} \left[\frac{\partial}{\partial T_{ij}}(\det - 1)\right](e) = 0$

Sl_2? $\sum a_{ij} \frac{\partial}{\partial T_{ij}}(T_{11}T_{22} - T_{12}T_{21} - 1)(e) = 0$

$= a_{11} T_{22}(e) + a_{22} T_{11}(e) = \underbrace{a_{11} + a_{22}}_{tr(A)}$

It's 2×2 matrices of trace zero