Last fact from alg. geometry :-

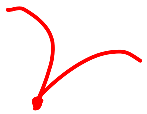
Theorem Let X be an irreducible affine variety. Then

$$\dim T_x(X) \geq \dim X \qquad \forall x \in X$$

Equality holds for all x in some dense open subset of X.

The x where $\dim T_x(X) = \dim X$ are called <u>simple points</u> of X. If ALL points are simple, then X is called <u>smooth</u>.

$V(T_1^2 - T_2^3) \subseteq \mathbb{A}^2$



Algebraic groups are smooth — every point is a translation of e.

So for an algebraic group G,

$$\boxed{\dim T_e(G) = \dim G^\circ}$$

Def Let G be an algebraic group. Define

$$L(G) := \{ D \in \mathrm{Der}(\Bbbk[G], \Bbbk[G]) \mid \lambda_g \circ D = D \circ \lambda_g \ \forall g \in G \}$$

"left invariant derivations"

$\lambda_g : \Bbbk[G] \to \Bbbk[G]$ "left translation by $g \in G$"

$$(\lambda_g f)(h) = f(g^{-1} h)$$

$\rho_g : \Bbbk[G] \to \Bbbk[G]$ "right translation"

$$(\rho_g f)(h) = f(hg)$$

$$\mathrm{Der}(\Bbbk[G], \Bbbk[G]) \underset{\text{Lie subalgebra}}{\leq} \mathfrak{gl}(\Bbbk[G])$$

Also $L(G) \leq \mathrm{Der}(\Bbbk[G], \Bbbk[G])$

$$\Big(\lambda_g \circ [D_1, D_2] = \lambda_g \circ (D_1 \circ D_2 - D_2 \circ D_1) = \cdots = [D_1, D_2] \circ \lambda_g\Big)$$

So $L(G)$ is a Lie algebra in very easy way.

Lemma | There's a linear isomorphism $L(G) \xrightarrow{\sim} T_e(G)$, $D \mapsto \overline{D} = ev_e \circ D$.

The inverse isomorphism is $T_e(G) \longrightarrow L(G)$, $X \mapsto \widetilde{X} = (id \mathbin{\overline{\otimes}} X) \circ m^*$.

$\overline{\otimes}$ means "contract"

$\Bbbk[G] \mathbin{\overline{\otimes}} \Bbbk \xrightarrow{\sim} \Bbbk[G]$

comultiplication

<u>Proof</u>

$$\tilde{X}(\underset{\substack{\uparrow \\ \Bbbk[G]}}{f})(\underset{\substack{\uparrow \\ G}}{g}) = (id \,\bar{\otimes}\, X)(m^*(f))(g) = \sum_{i=1}^{n} X(f_i'')\, f_i'(g)$$

$$= X\left(\sum_{i=1}^{n} f_i'(g)\, f_i''\right) = \underset{(\dagger)}{X(\lambda_{g^{-1}} f)}$$

$$m^*(f) = \sum_{i=1}^{n} f_i' \otimes f_i''$$

$$(\lambda_{g^{-1}} f)(h) = f(gh) = (m^* f)((g,h)) = \sum_{i=1}^{n} f_i'(g)\, f_i''(h)$$

Using this, let's check that $\tilde{X}$ is a <u>left invariant</u> <u>derivation</u>

$$\tilde{X}(\lambda_g f)(h) \overset{?}{=} \lambda_g \tilde{X}(f)(h)$$

$$\text{RHS} = \tilde{X}(f)(g^{-1}h) \overset{(\dagger)}{=} X(\lambda_{(g^{-1}h)^{-1}} f)$$

$$= X(\lambda_{h^{-1}} \lambda_g f)$$

$$\overset{(\dagger)}{=} \tilde{X}(\lambda_g f)(h) = \text{LHS} \ ✓$$

$$\tilde{X}(f_1 f_2)(g) = [\tilde{X}(f_1)(g)] f_2(g) + f_1(g) [\tilde{X}(f_2)(g)]$$

$$\text{LHS} \overset{(\dagger)}{=} X(\lambda_{g^{-1}}(f_1 f_2)) = X((\lambda_{g^{-1}} f_1)(\lambda_{g^{-1}} f_2))$$

$$= X(\lambda_{g^{-1}} f_1)(\lambda_{g^{-1}} f_2)(e) + (\lambda_{g^{-1}} f_1)(e)\, X(\lambda_{g^{-1}} f_2)$$

$$= [\tilde{X}(f_1)(g)] f_2(g) + f_1(g) [\tilde{X}(f_2)(g)]$$

$$= \text{RHS} \ ✓$$

Remains to check $\overline{\tilde{D}} = D$, $\tilde{\overline{X}} = X$

$$\tilde{\overline{D}}(f)(g) = D(f)(g)$$

$$\begin{aligned}
LHS &\overset{(+)}{=} \overline{D}(\lambda_{g^{-1}} f) \\
&= ev_e(D(\lambda_{g^{-1}} f)) \\
&= \lambda_{g^{-1}}(D(f))(e) \\
&= D(f)(g) = RHS \checkmark
\end{aligned}$$

$$\overline{\tilde{X}}(f) = X(f)$$

$$\begin{aligned}
LHS &= \tilde{X}(f)(e) \\
&\overset{(+)}{=} X(\lambda_{e^{-1}} f) \\
&= X(f) = RHS \checkmark
\end{aligned}$$

//

So: $L(G) \xrightarrow{\sim} T_e(G)$

$$D \longmapsto \overline{D}$$

$$\tilde{X} \longleftarrow X$$

$$\boxed{[X,Y] = ev_e \circ [\tilde{X}, \tilde{Y}]}$$

Using this, transport Lie algebra structure to make $T_e(G)$ into a Lie algebra too

Example $G = GL_n(\Bbbk)$ $\mathfrak{gl}_n(\Bbbk)$

What's the Lie bracket on $M_n(\Bbbk)$?

$$T_e(G) = M_n(\Bbbk)$$
$$\quad\quad\quad\ \ \rotatebox{90}{$\in$}$$
$$ev_e \circ \frac{\partial}{\partial T_{ij}} = e_{ij}$$

Expecting: $[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}$

Show: $[\tilde{e}_{ij}, \tilde{e}_{kl}](T_{rs}) = (\delta_{jk}\tilde{e}_{il} - \delta_{li}\tilde{e}_{kj})(T_{rs})$

What's $\tilde{e}_{ij}$? $\tilde{e}_{ij}(T_{rs}) = (id \,\bar{\otimes}\, e_{ij})(m^*(T_{rs})) = (id \,\bar{\otimes}\, e_{ij})\left(\sum_{u=1}^{n} T_{ru} \otimes T_{us}\right)$

$$= \sum_{u=1}^{n} e_{ij}(T_{us})\, T_{ru} \underset{i=u,\, j=s}{=} \delta_{js}\, T_{ri}$$

$$= \delta_{jk}\delta_{ls} T_{ri} - \delta_{li}\delta_{js} T_{rk}$$

$RHS = \delta_{jk}\tilde{e}_{il}(T_{rs}) - \delta_{li}\tilde{e}_{kj}(T_{rs})$

$LHS = \tilde{e}_{ij}(\tilde{e}_{kl}(T_{rs})) - \tilde{e}_{kl}(\tilde{e}_{ij}(T_{rs})) = \tilde{e}_{ij}(\delta_{ls} T_{rk}) - \tilde{e}_{kl}(\delta_{js} T_{ri})$

$= \delta_{rs}\delta_{jk} T_{ri} - \delta_{js}\delta_{li} T_{rk} = RHS$ ✓

$$L(GL_n(\Bbbk)) = \mathfrak{gl}_n(\Bbbk)$$

Lemma 2 Let $\varphi : G \to H$ be a morphism of algebraic groups.

Then $d\varphi_e : T_e(G) \to T_e(H)$ is a Lie algebra homomorphism.

$\Longrightarrow$ there's a functor from alg. gps to f.d. Lie algebras.

Proof

$$\begin{tikzcd} L(G) \arrow[r, "d\varphi"] \arrow[d, "S"', "D \mapsto ev_{e_G} \circ D"] \arrow[dr, phantom, "\circ"] & L(H) \arrow[d, "S", "D \mapsto ev_{e_H} \circ D = ev_{e_G} \circ \varphi^* \circ D"'] \\ T_e(G) \arrow[r, "d\varphi_e"'] & T_e(H) \end{tikzcd}$$

Let $d\varphi : L(G) \to L(H)$ be unique linear map so diagram commutes.

Claim For $D \in L(G)$, $(d\varphi)(D)$ is the unique element of $L(H)$ such that

$$\boxed{D \circ \varphi^* = \varphi^* \circ (d\varphi)(D)} \quad (*)$$

Pf. It's determined by $ev_{e_G} \circ D \circ \varphi^* = ev_{e_G} \circ \varphi^* \circ (d\varphi(D))$

How to get from here to $(*)$? Compose on right $\lambda_{\varphi(g)}$ for $g \in G$.

$g \in G$

$$ev_{e_G} \circ D \circ \varphi^* \circ \lambda_{\varphi(g)} = ev_{e_G} \circ \varphi^* \circ (d\varphi(D)) \circ \lambda_{\varphi(g)}$$

$$\varphi^* \circ \lambda_{\varphi(g)} = \lambda_g \circ \varphi^*$$

Exercise !

$$ev_g \circ D \circ \varphi^* = ev_g \circ \varphi^* \circ (d\varphi(D))$$

$$ev_{e_G} \circ \lambda_g = ev_g$$

Obvious !

This holds $\forall g \in G$, hence

$$D \circ \varphi^* = \varphi^* \circ (d\varphi(D))$$

Proves the claim //

Now to prove the lemma: $D_1, D_2 \in L(G)$

$$d\varphi([D_1, D_2]) = [d\varphi(D_1), d\varphi(D_2)]$$

LHS is unique element of $L(H)$ such that $\varphi^* \circ d\varphi([D_1, D_2]) = [D_1, D_2] \circ \varphi^*$

So we must check RHS has this property: $\varphi^* \circ [d\varphi(D_1), d\varphi(D_2)] = [D_1, D_2] \circ \varphi^*$

lhs $\varphi^* \circ d\varphi(D_1) \circ d\varphi(D_2) - \varphi^* \circ d\varphi(D_2) \circ d\varphi(D_1) \overset{(*)}{=} D_1 \circ D_2 \circ \varphi^* - D_2 \circ D_1 \circ \varphi^*$ ✓ //