From now on I'll identify $L(G) \equiv T_e(G)$, it's a Lie alg.!

$\varphi : G \to H$ morphism of algebraic groups $\Bigg\}$ $d\varphi : L(G) \to L(H)$, it's a Lie alg. homomorphism!

$\overset{\text{III}}{d\varphi_e}$

We have a functor $(\text{alg.gps.}) \longrightarrow (\text{f.d. Lie algs.})$

$$G \longmapsto L(G)$$
$$\varphi \longmapsto d\varphi$$

in terms of left invariant derivation $X\, I(H) \subseteq I(H)$

Final gimmic: For a group G, I'll often write simply $\mathfrak{g}$

- $\dim \mathfrak{g} = \dim G^\circ$
- if $H \leq G$ closed subgroup then $\mathfrak{h} \equiv \{X \in \mathfrak{g} \mid X\, I(H) = 0\}$
- $L(GL_n(\mathbb{k})) \equiv \mathfrak{gl}_n(\mathbb{k})$ so $e_{ij} \equiv ev_e \circ \dfrac{\partial}{\partial T_{ij}}$

general linear Lie algebra

Examples

① $L(SL_n(\mathbb{k})) = \mathfrak{sl}_n(\mathbb{k}) \leq \mathfrak{gl}_n(\mathbb{k})$, trace zero matrices

$$1 \to SL_n(\mathbb{k}) \to GL_n(\mathbb{k}) \xrightarrow{\det} \mathbb{G}_m \to 1$$

differentiate

$$0 \to \mathfrak{sl}_n(\mathbb{k}) \to \mathfrak{gl}_n(\mathbb{k}) \xrightarrow{tr} \mathbb{k} \to 1$$

$[x,x]=0$ Abelian

② $L(Sp_{2n}(\mathbb{k})) = \mathfrak{sp}_{2n}(\mathbb{k}) \leq \mathfrak{gl}_{2n}(\mathbb{k})$

What subspace? The symplectic Lie algebra.

$$\underset{\text{Gram matrix}}{J} = \left(\begin{array}{c|c} 0 & J_n \\ \hline -J_n & 0 \end{array}\right) \quad \text{where} \quad J_n = \begin{pmatrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{pmatrix}_{n \times n}$$

$Sp_{2n}(\mathbb{k})$ is $g \in GL_{2n}(\mathbb{k})$ s.t. $g^T J g = J$.

From $g^T J g = J$, we get generators for $I(Sp_{2n}(\mathbb{k})) \trianglelefteq \mathbb{k}[GL_{2n}(\mathbb{k})]$

$$I = \left\langle \sum_{r,s=1}^{2n} T_{ru} J_{rs} T_{sv} - J_{uv} \;\middle|\; \forall u,v = 1,\dots,2n \right\rangle$$

So $sp_{2n}(\mathbb{k})$ is all $A = \sum_{i,j=1}^{2n} a_{ij} e_{ij} \in gl_{2n}(\mathbb{k})$ such that

$$\sum_{i,j=1}^{2n} a_{ij} \frac{\partial}{\partial T_{ij}} \left(\sum_{r,s=1}^{2n} T_{ru} J_{rs} T_{sv} - J_{uv} \right)(e) = 0 \qquad \forall u,v$$

$$\sum_{i=1}^{2n} a_{iu} J_{iv} + \sum_{i=1}^{2n} J_{ui} a_{iv} = 0 \qquad \forall u,v$$

So $sp_{2n}(\mathbb{k}) = \{ A \in gl_{2n}(\mathbb{k}) \mid A^T J + JA = 0 \}$

$Sp(V) = \{ g \in GL(V) \mid (gv, gv') = (v, v') \quad \forall v,v' \in V \}$

$sp(V) = \{ A \in gl(V) \mid (Xv, v') + (v, Xv') = 0 \quad \forall v,v' \in V \}$

$$\dim Sp_{2n}(k) = \dim sp_{2n}(k)$$ ← easy to compute as its just linear algebra

$$\begin{pmatrix} A & \cancel{B} \\ \cancel{C} & \cancel{D} \end{pmatrix} \in sp_{2n}(k) \qquad -A^{\dagger}$$

$$\begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\begin{pmatrix} -C^T J_n & A^T J_n \\ -D^T J_n & B^T J_n \end{pmatrix} = \begin{pmatrix} -J_n C & -J_n D \\ J_n A & J_n B \end{pmatrix}$$

$$2 \cdot \frac{1}{2} n(n+1) + n^2$$

$$B = \underbrace{J_n B^T J_n}_{} = B^{\dagger}, \quad C = C^{\dagger}, \quad D = -A^{\dagger}$$

$$B^{\dagger} = \left(\quad \right)$$ B flipped in odd diagonal

$$\dim = n(n+1) + n^2 = \boxed{2n^2 + n}$$

③ Let G be any algebraic group, $g \in G$

$$\operatorname{Int} g : G \longrightarrow G \qquad \text{Inner automorphism}$$
$$x \longmapsto g x g^{-1}$$

$$\operatorname{Ad} g := d(\operatorname{Int} g) : \mathfrak{g} \longrightarrow \mathfrak{g}$$
$$X \longmapsto (\operatorname{Ad} g)(X)$$

$\mathfrak{gl}_n(\mathbb{k})$

$\in$

For $G = GL_n(\mathbb{k})$, $(\operatorname{Ad} g)(X) = g X g^{-1}$

Proof
$$\underbrace{((\operatorname{Ad} g)(e_{ij}))}_{\substack{n\times n \text{ matrix with } rs\text{-entry} \\ [g e_{ij} g^{-1}]_{rs}}}(T_{rs}) = e_{ij}\Big(\underbrace{(\operatorname{Int} g)^*(T_{rs})}\Big)$$

$$e_{ij}\left(\sum_{a,b=1}^{n} T_{ra}(g)\, T_{ab}\, T_{bs}(g^{-1})\right)$$

$$= T_{ri}(g)\, T_{js}(g^{-1})$$

//

(4) $\mathrm{Int} : G \longrightarrow \mathrm{Aut}_{gps}(G)$

$g \longmapsto \mathrm{Int}\, g$

NB $\mathrm{Aut}_{gps}(G)$ is NOT an algebraic group in general.

eg $G = \mathbb{G}_m \times \mathbb{G}_m$

group homomorphism

NOT a morphism of algebraic group

$\mathrm{Ad} : G \longrightarrow \mathrm{Aut}_{Lie}(\mathfrak{g}) \leq GL(\mathfrak{g})$

$g \longmapsto \mathrm{Ad}\, g$

closed subgroup of $GL(\mathfrak{g})$, so an alg. group.

Ad is a morphism of algebraic groups

Lie alg. of derivations $\mathfrak{g} \to \mathfrak{g}$ is $L(\mathrm{Aut}_{Lie}(\mathfrak{g}))$

$\mathrm{ad} := d(\mathrm{Ad}) : \mathfrak{g} \longrightarrow \mathrm{Der}(\mathfrak{g},\mathfrak{g}) \leq \mathfrak{gl}(\mathfrak{g})$

$X \longmapsto \mathrm{ad}\, X$

normal subgp.

$\mathrm{Int}(G) \trianglelefteq \mathrm{Aut}(G)$

Painful proof!

Omit calculation:

Do GL_n ...

$(\mathrm{ad}\, X)(Y) = [X, Y]$

$\mathrm{ad}(\mathfrak{g}) \underset{\text{ideal}}{\trianglelefteq} \mathrm{Der}(\mathfrak{g})$

inner derivations