$$\mathrm{Int} : G \longrightarrow \mathrm{Aut}(G)$$
$$g \longmapsto (x \mapsto gxg^{-1})$$

$$\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(\mathfrak{g}) \leq GL(\mathfrak{g})$$ "adjoint representation" of G

$\mathrm{Ad}\, g := d(\mathrm{Int}\, g)$

For $G = GL_n(\mathbb{k})$, $(\mathrm{Ad}\, g)(X) = gXg^{-1}$

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{Der}(\mathfrak{g}) \leq \mathfrak{gl}(\mathfrak{g})$$
$$X \longmapsto (Y \mapsto [X,Y])$$

$\mathrm{ad} = d(\mathrm{Ad})$

eg $G = SL_2(\mathbb{k})$ $\quad \mathfrak{g} = \mathfrak{sl}_2(\mathbb{k})$ 3-D 2x2 trace zero matrices

$\mathrm{Ad} : SL_2(\mathbb{k}) \longrightarrow GL(\mathfrak{sl}_2(\mathbb{k}))$ morphism of alg. gps.

$\ker \mathrm{Ad} = \{\pm I_2\} = Z(G)$

$\mathrm{Im}\, \mathrm{Ad} =: PSL_2(\mathbb{k})$ (a closed subgroup of $GL(\mathfrak{g})$)

$p \neq 2$ $\quad PSL_2(\mathbb{k}) \cong SL_2(\mathbb{k}) / Z$ as alg. gps.

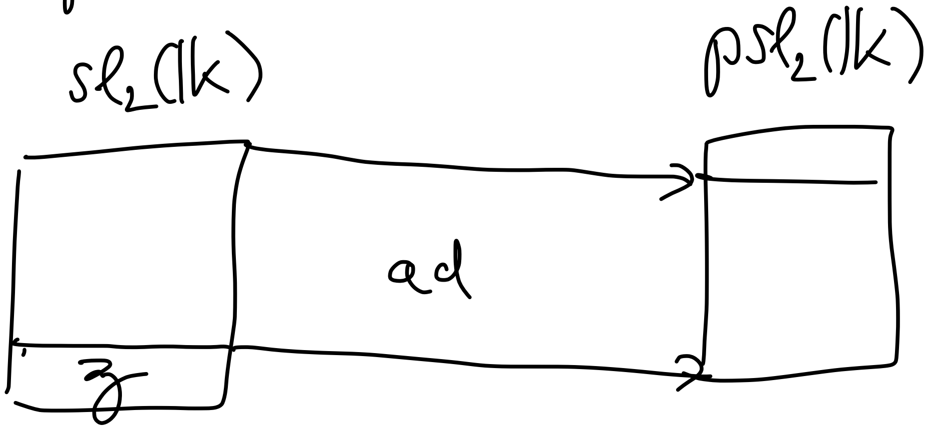
$p = 2$ $\quad Z = 1$ $\quad PSL_2(\mathbb{k}) \cong SL_2(\mathbb{k})$ as abstract groups but NOT as alg. gps.

$$ad : \mathfrak{sl}_2(\mathbb{k}) \longrightarrow \mathfrak{psl}_2(\mathbb{k}) = L(PSL_2(\mathbb{k}))$$

ker ad is 0 when $p \neq 2$ (so ad is an $\cong$)

But it is 1-D $\left\{ \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \right\}$ when $p = 2$ trace zero!!

So when $p=2$, ad is not an $\cong$, and Ad not an $\cong$ of varieties



Not an $\cong$.

$\mathfrak{z}$ is 1-D

$\mathfrak{z} = 0$ ← Why??

$\mathfrak{z}$ = center

$p \neq 2$ $\mathfrak{sl}_2(\mathbb{k}) \cong \mathfrak{psl}_2(\mathbb{k})$

$p = 2$ $\mathfrak{sl}_2(\mathbb{k}) \not\cong \mathfrak{psl}_2(\mathbb{k})$

Let's calculate $\mathfrak{psl}_2(\mathbb{k})$ explicitly.

$$Ad : SL_2(\mathbb{k}) \longrightarrow GL_3(\mathbb{k})$$

Use basis $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{k})$

$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$\left(Ad \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{ad-bc=1} \right)(e) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} d & -b \\ -a & a \end{pmatrix}$$

$$= \begin{pmatrix} -ac & a^2 \\ -c^2 & ac \end{pmatrix}$$

$$\therefore \; Ad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{array}{c} \begin{matrix} e \quad & \quad h \quad & \quad f \end{matrix} \\ \begin{pmatrix} a^2 & -2ab & -b^2 \\ -ac & ad+bc & bd \\ -c^2 & 2cd & d^2 \end{pmatrix} \end{array}$$

This matrix $\forall a,b,c,d \in \mathbb{k}$ with $ad-bc=1$ give $PSL_2(\mathbb{k}) < GL_3(\mathbb{k})$

Now $p=2$

$$\begin{pmatrix} a^2 & 0 & b^2 \\ ac & 1 & bd \\ c^2 & 0 & d^2 \end{pmatrix}$$

equations

$T_{21}^2 = T_{11} T_{31}$, $T_{23}^2 = T_{13} T_{33}$

$T_{12} = T_{32} = 0$, $T_{22} = 1$

$T_{11} T_{33} + T_{13} T_{31} = 1$

$$\Longrightarrow \quad \mathfrak{psl}_2(\mathbb{k}) = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 0 & \gamma \\ 0 & 0 & -\alpha \end{pmatrix} \right\} < \mathfrak{gl}_3(\mathbb{k})$$

For rest of course, going to work in characteristic zero only.

$\cancel{\mathbb{K}}$ $\quad \mathbb{C}$

$\Downarrow$

ALL MORPHISMS ARE SEPARABLE

If G is an alg. group and H is a closed subgroup, there's a way to make set G/H of left cosets of H in G into a variety.

affine when $H \trianglelefteq G$
but in general quasi-projective

This has universal property:

open morphism of varieties

$$\begin{tikzcd} G \arrow[r, "p"] \arrow[d, "f"'] & G/H \arrow[dl, dashed, "\exists!\, \bar{f}"] \\ X & \end{tikzcd} \qquad g \longmapsto gH$$

$dp_e : \mathfrak{g} \twoheadrightarrow T_{eH}(G/H)$ has kernel $\mathfrak{h}$, so that $T_{eH}(G/H) \cong \mathfrak{g}/\mathfrak{h}$

$\exists!\, \bar{f}$ morphism, an isomorphism of varieties if p is separable

morphism f whose fibers are the cosets of H in G

any other variety

$\therefore$ In char. 0, $\bar{f}$ is an $\cong$ of varieties.

Consequences $G \xrightarrow{\varphi} H$ morphism of algebraic group

(i)
$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "p"'] & H \\ G/\ker\varphi \arrow[ur, "\exists\bar{\varphi}"'] & \end{tikzcd}$$

$\bar{\varphi} : G/\ker\varphi \xrightarrow{\sim} H$ is an iso. of algebraic groups when char. is zero.

$$\boxed{L(\ker\varphi) = \ker d\varphi}$$

$$L(\ker p) = \ker dp$$

We saw a counterex. in pos. char. earlier ... $Ad : SL_2(\mathbb{k}) \to PSL_2(\mathbb{k})$ in char.

(2) Suppose $H, K \leq G$ closed subgroups.

Then $\boxed{L(H \cap K) = L(H) \cap L(K)}$ assuming char. zero.

Proof

$$\begin{tikzcd} G \arrow[r, "p"] & G/K \\ H \arrow[u, hook] \arrow[r, "\bar{p}"] \arrow[dr, "q"'] & HK/K \arrow[u, hook] \\ & H/H\cap K \arrow[u, "\exists!\,\bar{q}"'] \end{tikzcd}$$

$\cong$ of varieties

$\ker dp_e = L(K)$

$\therefore \ker d\bar{p}_e = L(H) \cap L(K)$

$\ker dq_e = L(H \cap K)$

③ Let $\rho : G \longrightarrow GL(V)$ be a representation of G

$d\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ a representation of $\mathfrak{g}$.

Let $W \leq V$ be any subspace. Then

$$\boxed{L(N_G(W)) = \mathfrak{n}_{\mathfrak{g}}(W)}$$

$$N_G(W) = \{g \in G \mid \underbrace{g(W)}_{\rho(g)(W)} \subseteq W\} \qquad \mathfrak{n}_{\mathfrak{g}}(W) = \{X \in \mathfrak{g} \mid \underbrace{X(W)}_{(d\rho(X))(W)} \subseteq W\}$$

<u>Proof</u> Use quotient results above to reduce to case $G \leq GL(V)$

$$L(N_{GL(V)}(W)) = \mathfrak{n}_{\mathfrak{gl}(V)}(W)$$

$$L(N_G(W)) = L(G \cap N_{GL(V)}(W)) = \mathfrak{g} \cap \mathfrak{n}_{\mathfrak{gl}(V)}(W) = \mathfrak{n}_{\mathfrak{g}}(W)$$

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