Working over $\mathbb{C}$ from now on!

(1) $\varphi : G \to H$ $\quad L(\ker \varphi) = \ker d\varphi$

(2) $H, K \leq G$ $\quad L(H \cap K) = L(H) \cap L(K)$

(3) $\rho : G \to GL(V)$, $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ — representations

$G \circlearrowright V$ $\quad$ $\mathfrak{g} \circlearrowright V$

$g \cdot v = \rho(g)(v)$ $\quad$ $X \cdot v = (d\rho)(X)(v)$

If $W \leq V$ any subspace, then $L(N_G(W)) = n_{\mathfrak{g}}(W)$

<u>Consequence</u> If G is connected, then W is G-stable $\iff$ W is $\mathfrak{g}$-stable

$N_G(W) = G \iff n_{\mathfrak{g}}(W) = \mathfrak{g}$

G, $\mathfrak{g}$ same notion of "submodule"

<u>Hence</u> V is irreducible representation of G $\iff$ V is an irreducible of $\mathfrak{g}$.

__Theorem__ (Lattice correspondence) Let G be connected $/\mathbb{C}$.

By an __algebraic subalgebra__ of $\mathfrak{g} = L(G)$, mean a subalgebra $\mathfrak{h}$ such that $\mathfrak{h} = L(H)$ for some closed connected $H \leq G$.

There's an inclusion preserving bijection

$$\left\{ \begin{matrix} \text{closed connected subgroups} \\ \text{of } G \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{algebraic subalgebras} \\ \text{of } \mathfrak{g} \end{matrix} \right\}$$

$$H \longmapsto \mathfrak{h} = L(H)$$

This satisfies $L(H \cap K) = L(H) \cap L(K)$.

Moreover, $H \trianglelefteq G \iff \mathfrak{h} \trianglelefteq \mathfrak{g}$

normal

ideal : $[x, y] \in \mathfrak{h} \ \forall x \in \mathfrak{h}, y \in \mathfrak{g}$.

Proof The map is onto by definition.

To see it's 1-1, suppose $H, K \leq G$ satisfying $\mathcal{L}(H) = \mathcal{L}(K)$.

$$\mathcal{L}(H \cap K) = \mathcal{L}(H) \cap \mathcal{L}(K) = \mathcal{L}(H) = \mathcal{L}(K)$$

Shows $\dim (H \cap K)^\circ = \dim H = \dim K$.

As H & K are irreducible and $H \cap K$ is closed subset, this implies

$H \cap K = H = K$ ✓

Finally, need to show $H \trianglelefteq G \iff \mathfrak{h} \trianglelefteq \mathfrak{g}$.

Look at Adjoint action of G on $\mathfrak{g}$, adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$.

By property ③, G acting via Ad and $\mathfrak{g}$ acting via ad leave the same subspaces invariant, so $\mathfrak{h} \trianglelefteq \mathfrak{g}$ is equivalent to saying that $\mathfrak{h}$ is stable under $\mathrm{Ad}\, g$ $\forall g \in G$.

So we want to show $H \leq G$ closed, connected that

$(\mathrm{Int}\, g)(H) \subseteq H \iff (\mathrm{Ad}\, g)(\mathfrak{h}) \subseteq \mathfrak{h}$
$\forall g \in G \qquad \forall g \in G$

Let $K = gHg^{-1}$, $\mathfrak{k} = L(K)$

$$\mathrm{Int}\, g|_H : H \xrightarrow{\sim} K$$

$$\mathrm{Ad}\, g|_{\mathfrak{h}} : \mathfrak{h} \xrightarrow{\sim} \mathfrak{k}$$

We see that $H = K \iff \mathfrak{h} = \mathfrak{k}$ thanks to lattice correspondence already established.

This is the fundamental principle of Lie Theory!!!

These strong results over $\mathbb{C}$ allow groups to be studied via their Lie algebras

$$G \longmapsto \mathfrak{g}$$

Simple algebraic groups $/\mathbb{C}$ $\rightsquigarrow$ f.d. simple Lie algebras $/\mathbb{C}$

alg. gp. with no closed connected normal subgroups other than 1 and G

Lie algebra with no ideal other than 0, $\mathfrak{g}$.

It'll turn out from that that every simple Lie algebra $\mathfrak{g}$ comes from a G.

$\exists\, G$ s.t. $L(G) = \mathfrak{g}$

We're going to classify these!

"almost" classification of simple alg. gps.

$SL_n(\mathbb{C}) \not\cong PSL_n(\mathbb{C})$ both have same Lie algebra $\mathfrak{sl}_n(\mathbb{C})$...

One can study representations of G ← simple alg. gp. $\longrightarrow$ f.d. representations of $\mathfrak{g}$

submodule $\longleftrightarrow$ submodules

We'll study these!

Show every f.d. representation of $\mathfrak{g}$ is completely reducible (Weyl's theorem) + classify irreducible ones.

$\Longleftarrow$ Every f.d. rep. of $\mathfrak{g}$ comes from a representation of G (suitable choice of G)

$SL_n(\mathbb{C})$ all reps. are c.r.

① Establish Weyl's theorem for $\mathfrak{sl}_n(\mathbb{C})$

<u>algebra</u>

② $SU_n \subset SL_n(\mathbb{C})$

maximal compact subgroup.

dense

Maschke's theorem

$\frac{1}{|G|}\sum_{g\in G}\dots \qquad \int_G \dots$

<u>analysis</u>

③ Flag variety, flags in $\mathbb{C}^n$.

$\circlearrowright$

$G=SL_n(\mathbb{C})$ ... "G/B" $\quad B=\begin{pmatrix} & * \\ 0 & \end{pmatrix}$

Irreps come from suitable line bundles on G/B

$$H^0(G/B, \mathcal{L}_\lambda)$$

<u>geometry</u>

Ch. 1 Basic Theory of f.d. Lie algebras / $\mathbb{C}$

I've been using much of general setup of Lie algebras already.

$\mathfrak{g}$ f.d. Lie algebra
- subalgebra $\mathfrak{h} \leq \mathfrak{g}$
- ideals $\mathfrak{n} \trianglelefteq \mathfrak{g}$
- quotient Lie algebra $\mathfrak{g}/\mathfrak{n}$
- normalizer, centralizer $n_{\mathfrak{g}}(\mathfrak{h})$ $c_{\mathfrak{g}}(\mathfrak{h})$
- center $\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x,y]=0 \ \forall y \in \mathfrak{g}\}$ $= c_{\mathfrak{g}}(\mathfrak{g})$

...etc...

If $\mathfrak{g} = L(G)$, G connected
$\mathfrak{g}$ is solvable $\Leftrightarrow$ G is solvable

Solvable $\mathfrak{g}'$ = derived subalgebra,
- generated by all $[x,y]$ $\forall x,y \in \mathfrak{g}$.
- smallest ideal of $\mathfrak{g}$ so $\mathfrak{g}/\mathfrak{g}'$ is Abelian.

$\mathfrak{g}$ is solvable if the derived series
$$\mathfrak{g} \geq \mathfrak{g}' \geq \mathfrak{g}'' \geq \cdots$$
eventually gets to 0.

Nilpotent
$$0 \leq \mathfrak{z}(\mathfrak{g}) \leq \cdots$$ ascending central series.
$\mathfrak{g}$ is nilpotent if this eventually gets to $\mathfrak{g}$.

$\mathfrak{g}$ is nilpotent $\Updownarrow$ G is nilpotent

Def (1) $\mathfrak{g}$ is simple if no ideals other than 0 and $\mathfrak{g}$ itself.
$\mathfrak{g} \neq 0$

(2) $\mathfrak{g}$ is semisimple if it has no non-zero solvable ideals.

In fact "Lie theory" works best for semisimple groups / Lie algebra.
We're mainly going to focus on f.d. semisimple Lie algebras.

We'll see: $\mathfrak{g}$ is semisimple $\iff$ it's a $\oplus$ of simple Lie algebras