

Working over \mathbb{C} from now on!

$$\textcircled{1} \varphi: G \rightarrow H \quad L(\ker \varphi) = \ker d\varphi$$

$$\textcircled{2} H, K \leq G \quad L(H \cap K) = L(H) \cap L(K)$$

$$\textcircled{3} \left. \begin{array}{l} \rho: G \rightarrow GL(V) \\ d\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V) \end{array} \right\} \begin{array}{l} \text{representations} \\ G \curvearrowright V \quad \mathfrak{g} \curvearrowright V \\ g \cdot v = \rho(g)(v) \quad X \cdot v = (d\rho)(X)(v) \end{array}$$

$$\text{If } W \leq V \text{ any subspace, then } L(N_G(W)) = \mathfrak{n}_{\mathfrak{g}}(W)$$

Consequence \nearrow If G is connected, then W is G -stable \iff W is \mathfrak{g} -stable

G, \mathfrak{g} same notion of "submodule"

$$N_G(W) = G \iff \mathfrak{n}_{\mathfrak{g}}(W) = \mathfrak{g}$$

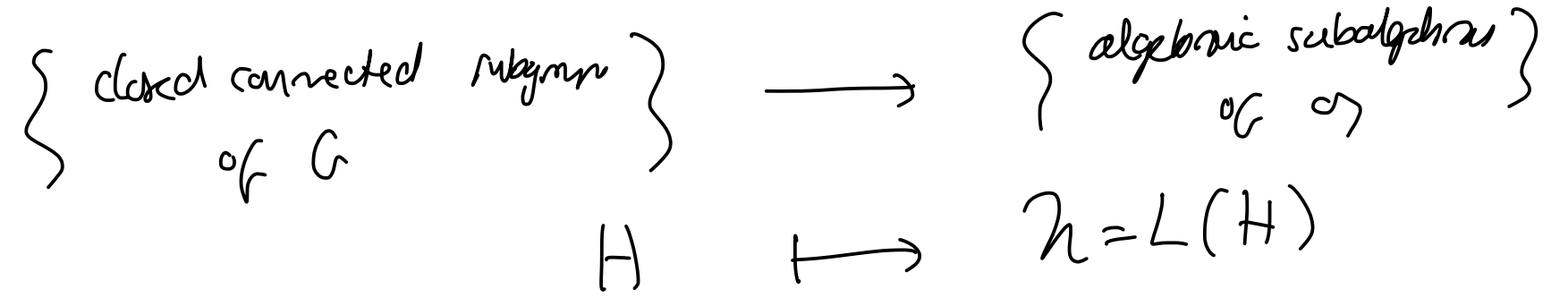
Hence V is irreducible representation of G

$\iff V$ is an irreducible of \mathfrak{g} .

Theorem (Lattice correspondence) Let G be connected $/ \mathbb{C}$.

By an algebraic subalgebra of $\mathfrak{g} = L(G)$, mean a subalgebra \mathfrak{h} such that $\mathfrak{h} = L(H)$ for some closed connected $H \leq G$.

There's an inclusion preserving bijection



This satisfies $L(H \cap K) = L(H) \cap L(K)$.

Moreover, $H \trianglelefteq G \iff \mathfrak{h} \trianglelefteq \mathfrak{g}$
↑ normal ↑ ideal : $[X, Y] \in \mathfrak{h} \quad \forall X \in \mathfrak{h}, Y \in \mathfrak{g}$.

Proof The map is onto by definition.

To see it's 1-1, suppose $H, K \leq G$ satisfying $L(H) = L(K)$.

$$L(H \cap K) = L(H) \cap L(K) = L(H) = L(K)$$

Shows $\text{der}(H \cap K)^0 = \text{der } H = \text{der } K$.

As H & K are indecomposable and $H \cap K$ is closed subset, this implies

$$H \cap K = H = K \quad \checkmark$$

Finally, need to show $H \trianglelefteq G \iff \mathfrak{h} \trianglelefteq \mathfrak{g}$.

Look at Adjoint action of G on \mathfrak{g} , adjoint action of \mathfrak{g} on \mathfrak{g} .

By property (3), G acting via Ad and \mathfrak{g} acting via ad leave the same subspace invariant, so $\mathfrak{h} \trianglelefteq \mathfrak{g}$ is equivalent to saying that

\mathfrak{h} is stable under $\text{Ad } g \quad \forall g \in G$.

So we want to show $H \leq G$ closed, connected that

$$(\text{Int } g)(H) \subseteq H \iff (\text{Ad } g)(\mathfrak{h}) \subseteq \mathfrak{h} \\ \forall g \in G \qquad \qquad \qquad \forall g \in G$$

$$\text{Let } K = gHg^{-1}, \quad \mathfrak{k} = L(K)$$

$$\text{Int } g|_H : H \xrightarrow{\sim} K$$

$$\text{Ad } g|_{\mathfrak{h}} : \mathfrak{h} \xrightarrow{\sim} \mathfrak{k}$$

We see that $H = K \iff \mathfrak{h} = \mathfrak{k}$ thanks to lattice
correspondence already established.

This is the fundamental principle of Lie Theory!!!

These strong results over \mathbb{C} allow groups to be studied via their Lie algebras

Simple algebraic groups G/\mathbb{C} $\xrightarrow{\quad}$ \mathfrak{g}
 $\xrightarrow{\quad}$ f.d. simple Lie algebras \mathfrak{g}

alg. gp. with no closed connected normal subgroups other than 1 and G

Lie algebra with no ideal other than $0, \mathfrak{g}$.

It'll turn out from that that every simple Lie algebra \mathfrak{g} comes from a G .

$\exists G$ s.t. $L(G) = \mathfrak{g}$

"almost" classification of simple alg. gps.

$SL_n(\mathbb{C}) \not\cong PSL_n(\mathbb{C})$ both have same Lie algebra $\mathfrak{sl}_n(\mathbb{C}) \dots$

We're going to classify these!

One can study representations of G ^{simple alg gp.} \longrightarrow f.d. representation of \mathfrak{g}
submodule \longleftarrow submodules

\uparrow
We'll study these!

Every f.d. rep. of \mathfrak{g} comes
from a representation of G
(suitable choice of G)



Show every f.d. representation
of \mathfrak{g} is completely reducible
(Weyl's Theorem)
+ classify irreducible ones.

$SL_n(\mathbb{C})$

all repr. are c.r.

①

Establish Weyl's theorem
for $sl_n(\mathbb{C})$
algebra

②

$SU_n \subset SL_n(\mathbb{C})$

maximal compact subgroup.

dense

Maschke's theorem

$$\frac{1}{|G|} \sum_{g \in G} \dots$$

$$\int_G \dots$$

analysis

③

Flag variety, flags in \mathbb{C}^n .

$G = SL_n(\mathbb{C}) \dots "G/B" \quad B = \begin{pmatrix} * & & \\ & \dots & \\ & & 0 \end{pmatrix}$

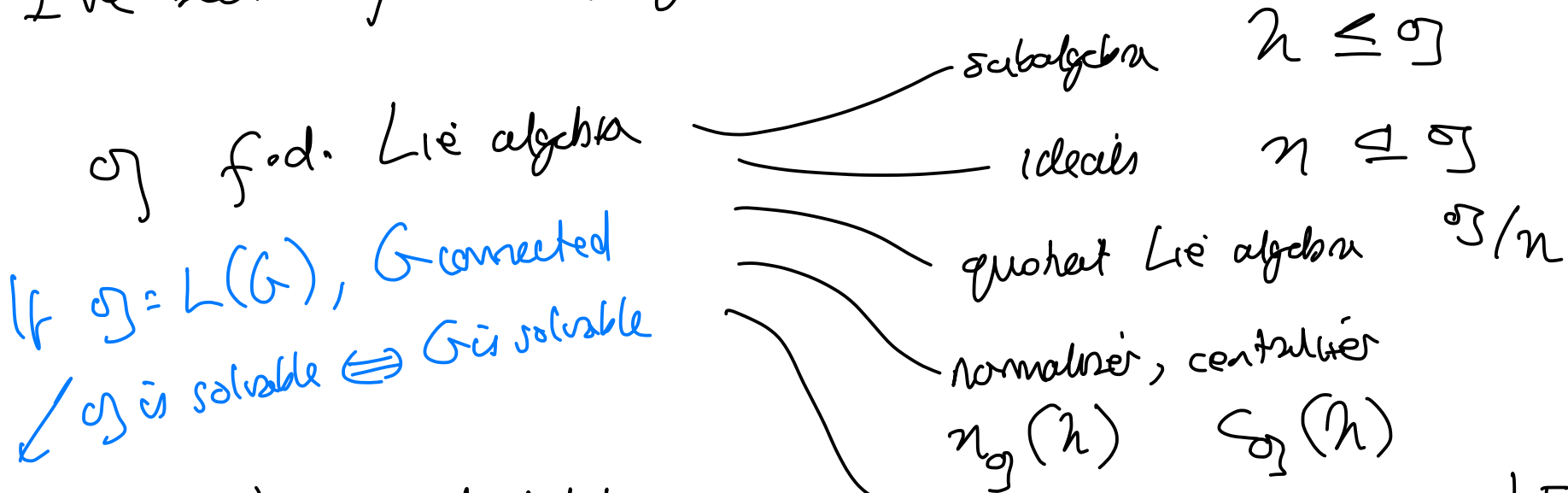
Irreps come from suitable line bundles on G/B

$$H^0(G/B, \mathcal{L}_\lambda)$$

geometry

Ch. 1 Basic Theory of f.d. Lie algebras (1)

I've been using much of general vector & Lie algebras already.



Solvable $\mathfrak{g}' =$ derived subalgebra,

- generated by all $[x, y] \ \forall x, y \in \mathfrak{g}$.
- smallest ideal of \mathfrak{g} so $\mathfrak{g}/\mathfrak{g}'$ is Abelian.

\mathfrak{g} is solvable if the derived series

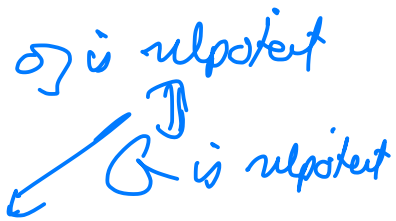
$$\mathfrak{g} \supseteq \mathfrak{g}' \supseteq \mathfrak{g}'' \supseteq \dots$$

eventually gets to 0.

Nilpotent

$$0 \leq z(\mathfrak{g}) \leq \dots \text{ ascending central series}$$

\mathfrak{g} is nilpotent if this eventually gets to \mathfrak{g} .



Def ① \mathfrak{g} is simple if no ideals other than $\{0\}$ and \mathfrak{g} itself.

② \mathfrak{g} is semisimple if it has no non-zero solvable ideals.

In fact "Lie Theory" works best for semisimple groups / Lie algebras.
We're mainly going to focus on f.d. semisimple Lie algebras.

We'll see: \mathfrak{g} is semisimple \iff it is a \oplus of simple Lie algebras