

\mathfrak{g} f.d. Lie algebra / \mathbb{C}

subalgebra ✓ ideals ✓ isomorphism theorems ✓
nilpotent ✓ solvable ✓ semisimple
(no non-zero solvable ideals)

first example $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ (Basis $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
Lie bracket $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$

Smallest (lowest dimensional) simple Lie algebra.

Claim This is a simple Lie algebra.

Stupid Proof Take $0 \neq ae + bh + cf$, let I be the ideal it generates,

show $I = \mathfrak{g}$.

$$[h, -] = 2ae - 2cf \in I$$

$$[h, [h, -]] = 4ae + 4cf \in I$$

$$\Rightarrow ae, cf, bh \in I$$

(if $a \neq 0$ then $[e, f] = h \in I$)

(if $c \neq 0$ then $[e, f] = h \in I$)

$\Rightarrow h \in I$, hence,

$$[h, e] = 2e \in I \quad \therefore e, h, f \in I$$

$$[h, f] = -2f \in I \quad \therefore I = \mathfrak{g}$$

You could try to classify all low-dimensional Lie algebras.

(1D) Abelian \mathbb{C}

(2D) Abelian $\mathbb{C} \oplus \mathbb{C}$
or basis e, h $[h, e] = ae + bh$, $a, b \in \mathbb{C}$ wlog $a \neq 0$.

$[h, [h, e]] = a[h, e]$... replace e by $[h, e]$ to reduce to the
situation that $[h, e] = ae$... replace h by $\frac{2}{a}h$ to reduce

to situation that $[h, e] = 2e$.

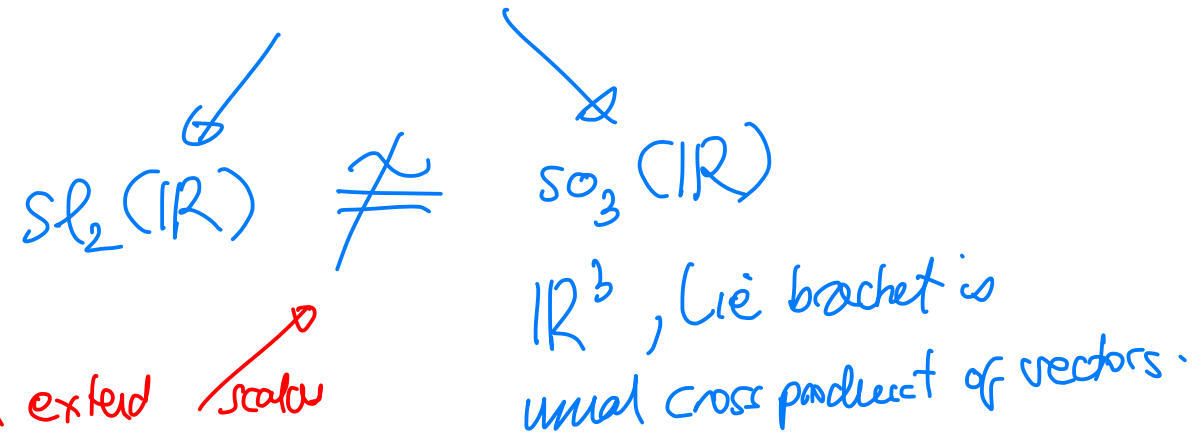
See that it is the 2D subalgebra of $\mathfrak{sl}_2(\mathbb{C})$ spanned by $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

This is a solvable Lie algebra, not nilpotent.

(3D)

- $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ Abelian } they are such
- $\mathbb{C} \oplus \mathbb{C}$ (the 2-D one just studied) $\mathbb{C} \ltimes (\mathbb{C} \oplus \mathbb{C})$
- A bunch of semidirect products $\mathfrak{g} \ltimes \mathfrak{n}$, need Lie alg. hom. $\mathfrak{g} \rightarrow \text{Der}(\mathfrak{n})$
(almost all are solvable, not nilpotent)
- Heisenberg Lie algebra, basis x, y, z , $[x, y] = z$, z is central
(nilpotent) $(\mathbb{C} \oplus \mathbb{C}) \ltimes \mathbb{C}$
- $sl_2(\mathbb{C})$ (simple)

Bianchi 1898.
 ↙
 Worked over \mathbb{R} not \mathbb{C}



When extend scalars from \mathbb{R} to \mathbb{C} , become \cong .

Next up More about representations of Lie algebras.

Def. Say V is a \mathfrak{g} -module if there's a bilinear map $\mathfrak{g} \times V \rightarrow V$
 $(x, v) \mapsto xv$
 such that $x(yv) - y(xv) = [x, y]v$.

If V is a \mathfrak{g} -module, get a Lie algebra homomorphism
 $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V), x \mapsto \rho(x)$ where $\rho(x)(v) = xv$

Conversely... so " \mathfrak{g} -module" \equiv "representation of \mathfrak{g} ".

Homomorphism of \mathfrak{g} -modules $f: V \rightarrow W$ linear map so $f(xv) = x f(v)$
 $\forall v \in V, x \in \mathfrak{g}$.

$$\begin{array}{ccc} V & \xrightarrow{\rho(x)} & V \\ f \downarrow & \circ & \downarrow f \\ W & \xrightarrow{\rho_W(x)} & W \end{array}$$



$\forall x \in \mathfrak{g}$.

Def. Let \mathfrak{g} be any Lie algebra

\mapsto universal enveloping algebra $U(\mathfrak{g})$ is the associative algebra

$$U(\mathfrak{g}) := T(\mathfrak{g}) / \langle x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g} \rangle$$

tensor algebra

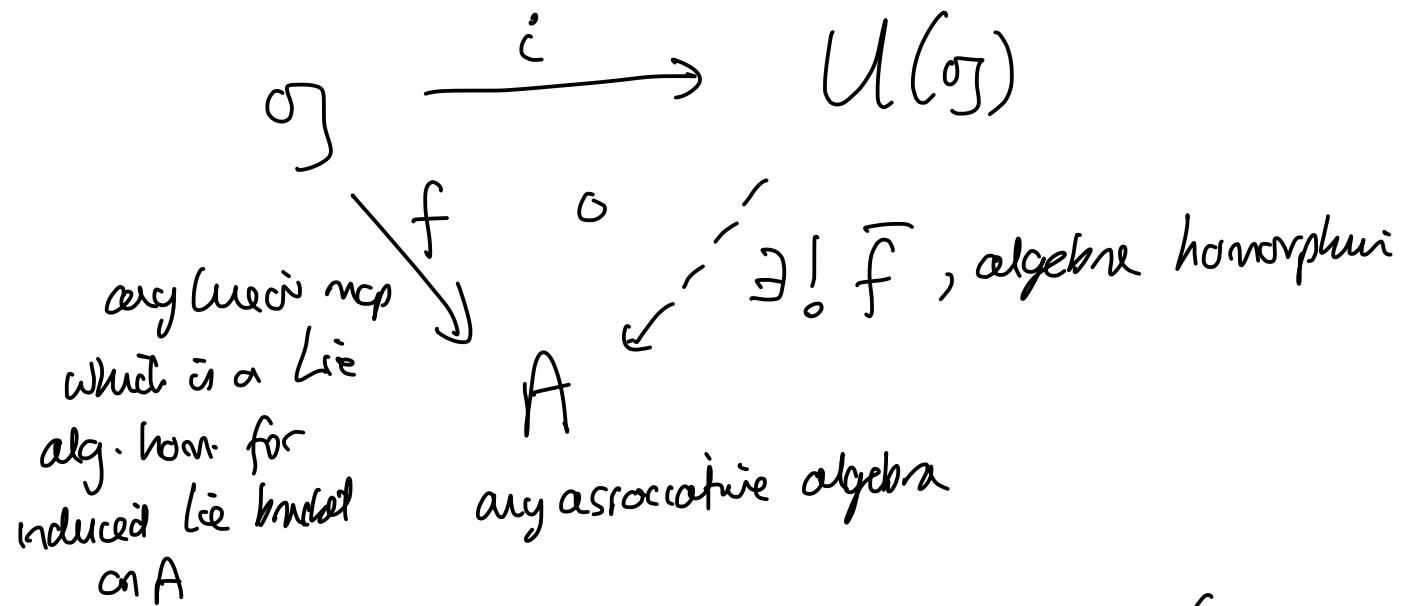
(free associative algebra on that vector space)

There's a linear map $i: \mathfrak{g} \rightarrow U(\mathfrak{g}), x \mapsto \bar{x}$.

\hookrightarrow actually a Lie algebra homomorphism if we view $U(\mathfrak{g})$ as a Lie algebra via the commutator

In fact, $U(\mathfrak{g})$ is the associative algebra which is universal

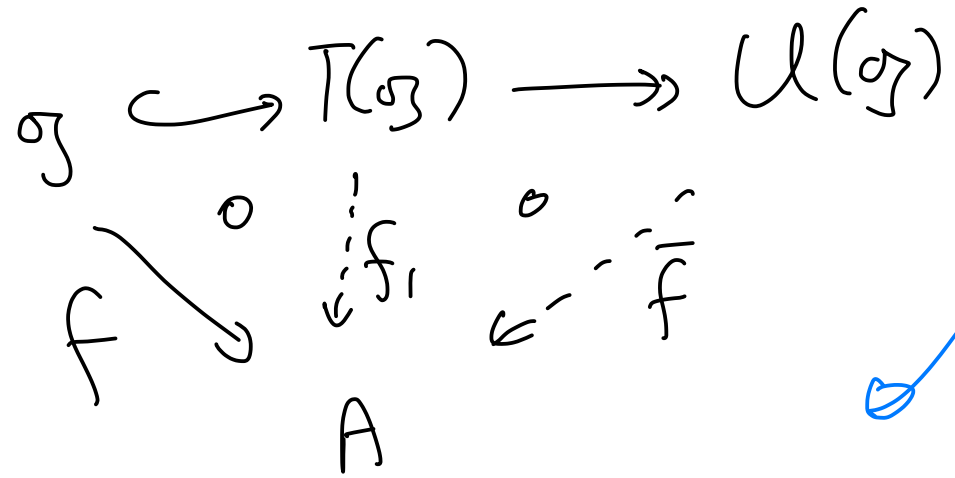
for this property...



In fact: this gives an isomorphism of categories

$(\mathfrak{g}\text{-modules}) \cong (\text{left } U(\mathfrak{g})\text{-modules})$

Proof



Lemma If V is a \mathfrak{g} -module, there's a unique way to make it into a module over the associative algebra $U(\mathfrak{g})$ so that $\bar{x}v = xv \forall x \in \mathfrak{g}$
 Conversely, any $U(\mathfrak{g})$ -module becomes a \mathfrak{g} -module via $xv = \bar{x}v \forall x \in \mathfrak{g}$

② Heisenberg x, y, z $[x, y] = z$, z central

$$U(\mathfrak{g}) = \frac{\mathbb{C}[z] \langle x, y \rangle}{\langle xy - yx - z \rangle}$$

set z to 1, you get

$$\frac{\mathbb{C} \langle x, y \rangle}{\langle xy - yx - 1 \rangle}$$

First Weyl algebra.

$\Delta = \text{coint}$, $\Sigma = \text{counit}$, $S = \text{antipode}$

G a finite group
 $\mathbb{C}G$ group algebra is a Hopf algebra
 $\Delta(g) = g \otimes g$
 $\Sigma(g) = 1$ $S(g) = g^{-1}$

G algebraic group
 $\mathbb{C}[G]$ coordinate algebra is a Hopf algebra
 $\Delta = m^*$
 $\Sigma = \text{ev}_e$ $S = i^*$

\mathfrak{g} Lie algebra, $\mathcal{U}(\mathfrak{g})$
 $\mathcal{U}(\mathfrak{g})$ univ. enveloping alg. is a Hopf algebra
 $\Delta(\bar{x}) = \bar{x} \otimes 1 + 1 \otimes \bar{x}$
 $\Sigma(\bar{x}) = 0$
 $S(\bar{x}) = -\bar{x}$

Proof:
 Use the univ. prop.

eg to construct Δ , you have to check

$$[\bar{x} \otimes 1 + 1 \otimes \bar{x}, \bar{y} \otimes 1 + 1 \otimes \bar{y}] = \overline{[x, y]} \otimes 1 + 1 \otimes \overline{[x, y]}.$$

(in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$)

$$\mathfrak{g} \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

$$x \longmapsto \bar{x} \otimes 1 + 1 \otimes \bar{x}$$

then use univ. property...