$\mathfrak{g}$ f.d. Lie algebra / $\mathbb{C}$

subalgebras ✓ ideals ✓ isomorphism theorems ✓

nilpotent ✓ solvable ✓ <u>semisimple</u>

(no non-zero solvable ideals)

<u>First example</u> $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$

Basis $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Lie bracket $[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$

Smallest (lowest dimensional) simple Lie algebra.

<u>Claim</u> This is a simple Lie algebra.

<u>Stupid Proof</u> Take $0 \neq ae + bh + cf$, let I be the ideal it generates, show $I = \mathfrak{g}$.

$[h, -] = 2ae - 2cf \in I$ $\quad [h,[h,-]] = 4ae + 4cf \in I$

$\Longrightarrow ae, cf, bh \in I$

If $a \neq 0$ then $[e,f] = h \in I$ $\quad$ If $c \neq 0$ then $[e,f] = h \in I$ $\quad \Rightarrow h \in I$, hence, $[h,e] = 2e \in I$, $[h,f] = -2f \in I$ $\quad \therefore e, h, f \in I$ $\quad \therefore I = \mathfrak{g}$

You could try to classify all low-dimensional Lie algebras.

(1D) Abelian $\mathbb{C}$

(2D) Abelian $\mathbb{C} \oplus \mathbb{C}$

<u>OR</u> basis e, h $\quad [h,e] = ae + bh$, $a, b \in \mathbb{C}$ WLOG $a \neq 0$.

$[h, [h,e]] = a[h,e]$ ... replace e by $[h,e]$ to reduce to the situation that $[h,e] = ae$ ... replace h by $\frac{2}{a}h$ to reduce to situation that $[h,e] = 2e$.

See that it's the 2D subalgebra of $\mathfrak{sl}_2(\mathbb{C})$ spanned by $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

This a <u>solvable</u> Lie algebra, not nilpotent.

(3D)

- $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ Abelian
- $\mathbb{C} \oplus$ (the 2-D one just studied) . $\mathbb{C} \ltimes (\mathbb{C} \oplus \mathbb{C})$

They are such

- A bunch of semidirect products (almost all are solvable, not nilpotent)

$\mathfrak{g} \ltimes \mathfrak{n}$, need Lie alg. hom. $\mathfrak{g} \to \mathrm{Der}(\mathfrak{n})$

- Heisenberg Lie algebra, basis x, y, z (nilpotent)

$[x, y] = z$, z is central

$(\mathbb{C} \oplus \mathbb{C}) \ltimes \mathbb{C}$

- $\mathfrak{sl}_2(\mathbb{C})$ (simple)

Bianchi 1898.

Worked over $\mathbb{R}$ not $\mathbb{C}$

$\mathfrak{sl}_2(\mathbb{R}) \not\cong \mathfrak{so}_3(\mathbb{R})$

$\mathbb{R}^3$, Lie bracket is usual cross product of vectors.

when extend scalar from $\mathbb{R}$ to $\mathbb{C}$, become $\cong$.

Next up More about representations of Lie algebras.

Def. Say V is a **$\mathfrak{g}$-module** if there's a bilinear map $\mathfrak{g} \times V \to V$, $(x,v) \mapsto xv$ such that
$$x(yv) - y(xv) = [x,y]v.$$

If V is a $\mathfrak{g}$-module, get a Lie algebra homomorphism
$$\rho : \mathfrak{g} \to \mathfrak{gl}(V), \quad x \mapsto \rho(x) \quad \text{where } \rho(x)(v) = xv$$

Conversely... so "$\mathfrak{g}$-module" $\equiv$ "representation of $\mathfrak{g}$".

Homomorphism of $\mathfrak{g}$-modules $f : V \to W$ linear map so $f(xv) = x f(v)$ $\forall v \in V, x \in \mathfrak{g}$.

$\Longleftrightarrow$

$$\begin{tikzcd}
V \arrow[r, "\rho_V(x)"] \arrow[d, "f"'] \arrow[dr, phantom, "\circ"] & V \arrow[d, "f"] \\
W \arrow[r, "\rho_W(x)"'] & W
\end{tikzcd}$$
$\forall x \in \mathfrak{g}$.

Def. Let $\mathfrak{g}$ be any Lie algebra

Its $\boxed{\text{universal enveloping algebra}}$ $U(\mathfrak{g})$ is the associative algebra

$$U(\mathfrak{g}) := T(\mathfrak{g}) \Big/ \langle x \otimes y - y \otimes x - [x,y] \mid x, y \in \mathfrak{g} \rangle$$

tensor algebra
(free associative algebra on that vector space)

There's a linear map $i : \mathfrak{g} \longrightarrow U(\mathfrak{g}),\quad x \longmapsto \overline{x}$.

Its actually a Lie algebra homomorphism if we view $U(\mathfrak{g})$ as a Lie algebra via the commutator

In fact, $U(\mathfrak{g})$ is the associative algebra which is universal for this property...

$$
\begin{tikzcd}
\mathfrak{g} \arrow[r, "i"] \arrow[dr, "f"'] \arrow[r, phantom, shift right=4ex, "\circ"] & U(\mathfrak{g}) \arrow[d, dashed, "\exists!\, \bar{f}"] \\
& A
\end{tikzcd}
$$

$\exists!\, \bar{f}$, algebra homomorphism

any linear map which is a Lie alg. hom. for induced Lie bracket on A

A: any associative algebra

In fact: this gives an isomorphism of categories

$$(\mathfrak{g}\text{-modules}) \cong (\text{left } U(\mathfrak{g})\text{-modules})$$

<u>Proof</u>

$$
\begin{tikzcd}
\mathfrak{g} \arrow[r, hook] \arrow[dr, "f"'] & T(\mathfrak{g}) \arrow[r, two heads] \arrow[d, dashed, "f_1"] & U(\mathfrak{g}) \arrow[dl, dashed, "\bar{f}"] \\
& A &
\end{tikzcd}
$$

//

<u>Lemma</u> If V is a $\mathfrak{g}$-module, there's a unique way to make it into a module over the associative algebra $U(\mathfrak{g})$ so that $\bar{x} v = x v \ \forall x \in \mathfrak{g}$.

Conversely, any $U(\mathfrak{g})$-module becomes a $\mathfrak{g}$-module via $x v = \bar{x} v \ \forall x \in \mathfrak{g}$

Examples

① $\mathfrak{g} = \mathbb{C}$, basis x, then $U(\mathfrak{g}) = \mathbb{C}[x]$

So understand $\mathfrak{g}$-modules in this case.

1-D $\mathfrak{g}$-modules $\longleftrightarrow$ points $\lambda \in \mathbb{C}$ (x acting as λ).

f.d. $\mathfrak{g}$-modules $\longleftrightarrow$ Jordan normal forms

indecomposable f.d. $\mathfrak{g}$-module $\longleftrightarrow$ Jordan block $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$

$\mathfrak{g} = L(\mathbb{G}_m)$ all reps. of $\mathbb{G}_m$ are c.r.

irreducibles 1-D $\lambda \in \mathbb{Z}$

$\|$

$L(\mathbb{G}_a)$

irreducible 1-D $\lambda = 0$.

② Heisenberg x, y, z $[x,y] = z$, z central

$$U(\mathfrak{g}) = \mathbb{C}[z]\langle x, y\rangle \Big/ \langle xy - yx - z\rangle$$

set z to 1, you get $\mathbb{C}\langle x,y\rangle \Big/ \langle xy - yx - 1\rangle$

First Weyl algebra.

Δ = comult., ε = counit, S = antipode

G a finite group	G algebraic group	$\mathfrak{g}$ Lie algebra, $x \in \mathfrak{g}$
$\mathbb{C}G$ group algebra is a Hopf algebra	$\mathbb{C}[G]$ coordinate algebra is a Hopf algebra	$U(\mathfrak{g})$ univ. enveloping alg. is a Hopf algebra
$\Delta(g) = g \otimes g$	$\Delta = m^*$	$\Delta(\bar{x}) = \bar{x} \otimes 1 + 1 \otimes \bar{x}$
$\varepsilon(g) = 1$ $\quad S(g) = g^{-1}$	$\varepsilon = ev_e$ $\quad S = i^*$	$\varepsilon(\bar{x}) = 0$
		$S(\bar{x}) = -\bar{x}$

Proof: Use the univ. prop.

eg to construct Δ, you have to check

$$[\bar{x} \otimes 1 + 1 \otimes \bar{x}, \bar{y} \otimes 1 + 1 \otimes \bar{y}] = \overline{[x,y]} \otimes 1 + 1 \otimes \overline{[x,y]}.$$

(in $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$)

$$\mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$$

$$x \longmapsto \bar{x} \otimes 1 + 1 \otimes \bar{x}$$

then use univ. property...